Warwick Mathematics Exchange

MA4A5

Algebraic Geometry

2025, May 1st

Desync, aka The Big Ree

Contents

Introduction

ree

Disclaimer: I make *absolutely no guarantee* that this document is complete nor without error. In particular, any content covered exclusively in lectures (if any) will not be recorded here. This document was written during the 2023 academic year, so any changes in the course since then may not be accurately reflected.

Notes on formatting

New terminology will be introduced in *italics* when used for the first time. Named theorems will also be introduced in *italics*. Important points will be **bold**. Common mistakes will be underlined. The latter two classifications are under my interpretation. YMMV.

Content not taught in the course will be outlined in the margins like this. Anything outlined like this is not examinable, but has been included as it may be helpful to know alternative methods to solve problems.

The table of contents above, and any inline references are all hyperlinked for your convenience.

History

First Edition: 2024-10-17*
Current Edition: 2024-10-17

Authors

This document was written by R.J. Kit L., a maths student. I am not otherwise affiliated with the university, and cannot help you with related matters.

Please send me a PM on Discord @Desync#6290, a message in the WMX server, or an email to Warwick.Mathematics.Exchange@gmail.com for any corrections. (If this document somehow manages to persist for more than a few years, these contact details might be out of date, depending on the maintainers. Please check the most recently updated version you can find.)

If you found this guide helpful and want to support me, you can buy me a coffee!

(Direct link for if hyperlinks are not supported on your device/reader: ko-fi.com/desync.)

*Storing dates in big-endian format is clearly the superior option, as sorting dates lexicographically will also sort dates chronologically, which is a property that little and middle-endian date formats do not share. See ISO-8601 for more details. This footnote was made by the computer science gang.

1 Review of Commutative Algebra

A *ring* $(R,+,\cdot,0_R,1_R)$, consists of a set R, two binary operations $+,\cdot : R\times R \to R$, and two distinguished elements $0_R,1_R \in R$ such that $(R,+)$ is an abelian group with identity 0_R; $(R,\cdot)$ is a monoid with identity 1_R; and multiplication distributes over addition.

All the rings we will consider will be commutative, unital, and non-trivial.

A function $f : R \to S$ between rings R and S is a *ring homomorphism* if for all $a,b \in R$,

(*i*) $f(a + b) = f(a) + f(b)$;

(*ii*) $f(ab) = f(a)f(b)$;

(*iii*) $f(1_R) = 1_S$.

An *ideal* I of a ring R is an additive subgroup that absorbs multiplication from the left (or equivalently for commutative rings, the right, or both sides), and we write $I \trianglelefteq R$ to denote this relation.

Example. The set I of polynomials with zero constant term is an ideal of $R = \mathbb{C}[x,y]$. Adding such polynomials has group structure since the coefficients all have additive inverses, and addition of polynomials is associative, commutative, and closed on I. Multiplying any polynomial in R by a polynomial in I yields another polynomial with zero constant term, so I also absorbs multiplication. △

Every element $x \in R$ generates an ideal $\langle x\rangle = xR = \{xr : r \in R\}$. An ideal of this form is called a *principal* ideal.

The *unit ideal* is the entire ring $R = \langle 1_R\rangle$, and the *zero* or *trivial ideal* is the set $\{0_R\} = \langle 0_R\rangle$. An ideal is *proper* if it is a proper subset of the ring; that is, it is not equal to the whole ring.

The intersection of arbitrary ideals is also an ideal, so we may define an ideal *generated by* a set $S \subseteq R$ by

$$\langle S\rangle = \bigcap_{\substack{S\subseteq I\subseteq R \\ I \text{ is an ideal}}} I$$

That is, the ideal $\langle S\rangle$ is then the smallest ideal containing S. We can also think of the ideal $\langle S\rangle$ as the collection of all finite R-linear combinations of elements of S.

An ideal I is *finitely generated* if there is a finite set S such that $I = \langle S\rangle = \langle s_1,\dots,s_n\rangle$.

Example. The elements of the ideal $I \trianglelefteq \mathbb{C}[x,y]$ of polynomials with zero constant term are of the form $xp(x,y) + yq(x,y)$, where $p,q \in \mathbb{C}[x,y]$. That is, every element is the $\mathbb{C}[x,y]$-linear combination of x and y, so $I = \langle x,y\rangle$. △

The preimage of an ideal under a ring homomorphism ϕ is an ideal. In particular, the *kernel* $\ker(\phi) = \phi^{-1}\big[\{0\}\big]$ is an ideal.

1.1 Special Elements, Rings, and Ideals

Let R be a commutative ring.

(*i*) An element $x \in R$ is a *unit* if $xy = 1$ for some $y \in R$ – in this case, y is uniquely determined by x and is also denoted by x^{-1};

(*ii*) An element $x \in R$ is a *zero-divisor* if $xy = 0$ for some $y \neq 0$;

(*iii*) An element $x \in R$ is *nilpotent* if $x^n = 0$ for some $n \geq 1$. (This also implies that x is zero-divisor, unless R is trivial.)

(*i*) R is a *field* if R is non-trivial and every non-zero element is a unit. In a field, the only ideals are the zero ideal and the unit ideal;

(*ii*) R is an *integral domain* if R is non-trivial and has no zero-divisors;

(*iii*) R is *reduced* if zero is the only nilpotent element.

(*i*) An ideal $\mathfrak{m} \subset R$ is *maximal* if the only ideal strictly containing it is the unit ideal R;

(*ii*) An ideal $\mathfrak{p} \subset R$ is *prime* if whenever $fg \in \mathfrak{p}$, we have $f \in \mathfrak{p}$ or $g \in \mathfrak{p}$;

(*iii*) An ideal $I \subset R$ is *radical* if whenever $x^n \in I$, $x \in I$.

The *radical* of an ideal I is the ideal

$$\sqrt{I} \coloneqq \{x \in R : \exists n > 0, x^n \in I\}$$

Equivalently, I is radical if $I = \sqrt{I}$.

Lemma 1.1. *Every maximal ideal is prime, and every prime ideal is radical.*

Theorem 1.2. *An ideal $I \trianglelefteq R$ is*

(*i*) *maximal;*

(*ii*) *prime;*

(*iii*) *radical,*

if and only if R/I is, respectively,

(*i*) *a field;*

(*ii*) *a integral domain;*

(*iii*) *a reduced ring.*

Given $I \trianglelefteq R$, the *quotient* R/I is the set of *cosets* $x+I = \{x+i : i \in I\}$. This quotient has ring structure under the addition and multiplication defined by $(x+I)+(y+I) = (x+y)+I$ (i.e. the inherited quotient group addition) and $(x+I)(y+I) = xy+I$. The quotient map $\pi : R \to R/I : x \mapsto x+I$ is then a surjective ring homomorphism with kernel I.

Because the quotient map π is a homomorphism, the preimage $\pi^{-1}[J]$ of any ideal $J \subseteq R/I$ is an ideal of R containing I. Conversely, π maps every ideal $K \subseteq R$ containing I onto an ideal in the quotient ring. Therefore, the ideals of R/I are in bijection with the ideals of R containing I:

$$\{\text{ideals of } R/I\} \cong \{\text{ideals of } R \text{ containing } I\}$$

This bijection carries maximal, prime, and radical ideals to maximal, prime, and radical ideals, respectively.

Theorem (First Isomorphism Theorem)**.** *Let $\phi : R \to S$ be a homomorphism with kernel I. Then, $R/I \cong \operatorname{im}(\phi)$. More precisely, the isomorphism $\bar{\phi} : R/I \to \operatorname{im}(\phi)$ is given by $\bar{\phi}(x+I) = \phi(x)$ for all $x \in R$.*

2 Affine Subvarieties

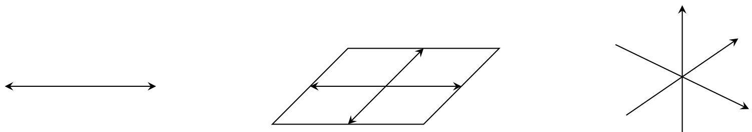
Let k be any field. Later, we will often assume that $k = \mathbb{C}$ since we will want to work over an algebraically closed field, but for now, we could also have $k = \mathbb{R}, \mathbb{Q}, \mathbb{Z}/p\mathbb{Z}$, etc. (In particular, the case where k is finite or p-adic field is of utility in number theory.)

The set $k^n = \big\{(x_1,x_2,\ldots,x_n) : x_i \in k\big\}$ is called *affine n-space* (*over k*), also denoted $\mathbb{A}^n_k$, or even just $\mathbb{A}^n$ if the field is clear or unimportant. We also sometimes write things like $\mathbb{A}^2_{x,y}$ to indicate the indeterminates.

Note that $\mathbb{A}^n_k$ is just k^n as a *set*; it is customary to use different notation since k^n is also a vector space over k, a ring, a topological space with the standard Euclidean topology, etc. We will write $\mathbb{A}^n_k$ whenever we wish to ignore this additional structure, or use an alternative (i.e. we will soon put a topology on $\mathbb{A}^n_k$ distinct from the standard topology on k^n).

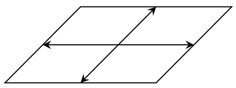
Example.

$k = \mathbb{R}$



$\mathbb{A}^1_\mathbb{R} = \mathbb{R}^1$ $\quad$ $\mathbb{A}^2_\mathbb{R} = \mathbb{R}^2$ $\quad$ $\mathbb{A}^3_\mathbb{R} = \mathbb{R}^3$

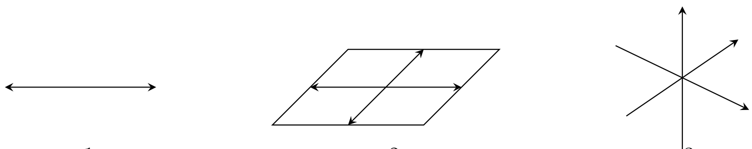
$k = \mathbb{C}$



$\mathbb{A}^1_\mathbb{C} = \mathbb{C}$

△

We can't draw $\mathbb{A}^4_\mathbb{R}$ or $\mathbb{A}^2_\mathbb{C}$ convincingly as they are 4-dimensional over $\mathbb{R}$, so we stop there. Later, we will define a notion of dimension specific to algebraic geometry where $\mathbb{A}^n$ is n-dimensional. Thus, we will suggestively choose to draw $\mathbb{A}^n_k$ as:



$\mathbb{A}^1_k$ $\quad$ $\mathbb{A}^2_k$ $\quad$ $\mathbb{A}^3_k$

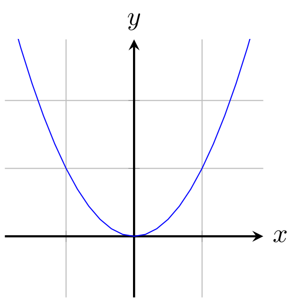
even if $k = \mathbb{C}$. In light of this, the set $\mathbb{A}^1_\mathbb{C} = \mathbb{C}^1$ is then called the *complex line*, while $\mathbb{A}^2_\mathbb{C} = \mathbb{C}^2$ is called the *complex plane*. (Contrast with analytic contexts, where "complex plane" often refers to $\mathbb{C}^1$, while the term "complex *coordinate* plane" is used for $\mathbb{C}^2$.)

Let $p \in k[x_1, \ldots, x_n]$ be a polynomial. Then, the *vanishing locus* or *zero locus* of p is the set of points upon which p vanishes:

$$\mathbb{V}(p) := \{x \in \mathbb{A}^n : p(x) = 0\}$$

Example. Consider the polynomial $y - x^2 \in \mathbb{C}[x,y]$. Then, the vanishing locus is the set:

$$\mathbb{V}(y - x^2) = \left\{(x,y) \in \mathbb{C}^2 : y = x^2\right\}$$



Note that this picture really depicts $\mathbb{V}(y - x^2) \cap \mathbb{R}^2$. △

When drawing a sketch of a vanishing locus V in $\mathbb{C}^n$, we will only draw its real points $V \cap \mathbb{R}^n$.

More generally, the vanishing locus of a set $S = \{f_i\}_{i \in I} \subseteq k[x_1, \ldots, x_n]$ of polynomials is the set of points upon which all the polynomials in S vanish:

$$\mathbb{V}(S) := \{x \in \mathbb{A}^n : \forall p \in S, p(x) = 0\}$$

If $S = \{f_1, \ldots, f_k\}$ is finite, then we also write $\mathbb{V}(S) = \mathbb{V}\big(\{f_1, \ldots, f_k\}\big)$ as $\mathbb{V}(f_1, \ldots, f_k)$.

Example. $\mathbb{V}(x,y) \subseteq \mathbb{C}^3$ is the complex line in $\mathbb{C}^3$ consisting of the z-axis. △

Theorem 2.1.

(*i*) *For any* $S_1, S_2 \subseteq k[x_1, \ldots, x_n]$,

$$\mathbb{V}(S_1) \cup \mathbb{V}(S_2) = \mathbb{V}(S_1 S_2)$$

where $S_1 S_2 = \{fg : f \in S_1, g \in S_2\}$

(*ii*) *If* I *is any set indexing a collection of sets* $S_i \subseteq k[x_1, \ldots, x_n]$ *of polynomials, then*

$$\bigcap_{i \in I} \mathbb{V}(S_i) = \mathbb{V}\left(\bigcup_{i \in I} S_i\right)$$

Proof.

(*i*) If $x \in \mathbb{V}(S_1) \cup \mathbb{V}(S_2)$, then $f(x) = 0$ for all $f \in S_1$, or $g(x) = 0$ for all $g \in S_2$. In either case, $(fg)(x) = f(x)g(x) = 0$ for all $f \in S_1$ and $g \in S_2$, so $x \in \mathbb{V}(S_1 S_2)$.

For the reverse containment, suppose that $x \notin \mathbb{V}(S_1) \cup \mathbb{V}(S_2)$, so there exist $f \in S_1$ and $g \in S_2$ such that $f(x) \neq 0$ and $g(x) \neq 0$. Then, $(fg)(x) \neq 0$, so $x \notin \mathbb{V}(S_1 S_2)$, proving the claim by contraposition.

(*ii*) We have $x \in \bigcap \mathbb{V}(S_i)$ if and only if x vanishes on every S_i. But this holds if and only if x vanishes on the union of the S_i, i.e., $x \in \mathbb{V}\left(\bigcup_{i \in I} S_i\right)$.

■

An *affine algebraic set* in $\mathbb{A}_k^n$ is the common vanishing locus of some collection $\{F_i\}_{i \in I}$ of polynomials in $k[x_1, \ldots, x_n]$. Note that the indexing set I may not necessarily be finite or even countable.

If k is algebraically closed, then an affine algebraic set of $\mathbb{A}_k^n$ is an (*affine*) *subvariety of* $\mathbb{A}_k^n$.

Example.

(*i*) The entire space $k^n = \mathbb{V}(0)$ is itself an affine algebraic set.

(*ii*) The empty set $\varnothing = \mathbb{V}(1)$ is an affine algebraic set.

(*iii*) Any point $a = (a_1, \ldots, a_n) \in \mathbb{A}^n$ is an affine algebraic set since $\{a\} = \mathbb{V}(x_1 - a_1, x_2 - a_2, \ldots, x_n - a_n) = \mathbb{V}\big(\{x_i - a_i\}_{i=1}^n\big)$.

(*iv*) Any finite subset $S \subseteq \mathbb{A}^n$ is also an affine algebraic set:

$$S = \bigcup_{s \in S} \{s\} = \bigcup_{s \in S} \mathbb{V}(x_i - s_i) = \mathbb{V}\left(\prod_{s \in S} (x_i - s_i)\right)$$

△

In fact, for $\mathbb{A}_k^1$, these are the only affine algebraic sets possible:

Lemma 2.2. *The affine algebraic sets of* $\mathbb{A}_k^1$ *are precisely* $\mathbb{A}_k^1$, $\varnothing$, *and all finite subsets.*

Proof. If f is the zero polynomial, then $\mathbb{V}(f) = \mathbb{A}^1$. Otherwise, f is some polynomial of degree d. Then, f has at most d roots, so $\mathbb{V}(f) = \{\text{roots of } f\}$ has cardinality at most d and is, in particular, finite. Then, for any collection $\{f_i\}_{i\in I}$,

$$\mathbb{V}\big(\{f_i\}_{i\in I}\big) = \bigcap_{i\in I} \mathbb{V}(f_i)$$

is the intersection of sets that are either all of $\mathbb{A}^1$, finite, or empty and is thus itself either all of $\mathbb{A}^1$, finite, or empty.

The converse statement that $\mathbb{A}^1$, $\varnothing$, and every finite subset is an affine algebraic set is shown in the previous example. ■

Example. Consider the set $S^1 = \big\{\cos(t) + i\sin(t) : t \in \mathbb{R}\big\} \subseteq \mathbb{A}^1_{\mathbb{C}}$. This set is infinite, but is not all of $\mathbb{A}^1_{\mathbb{C}}$, and is thus not a subvariety of $\mathbb{A}^1_{\mathbb{C}}$. △

A *hypersurface* is the vanishing locus $\mathbb{V}(f)$ of a single polynomial in $\mathbb{A}^n$. If $n = 2$, then such a vanishing locus is also called an *affine plane curve*.

Lemma 2.3. *The countably infinite union of affine algebraic sets is not necessarily an affine algebraic set.*

Proof. Consider $\mathbb{A}^1_{\mathbb{C}}$. For each integer $a \in \mathbb{Z}$, the singleton $\{a\} = \mathbb{V}(x - a)$ is a subvariety, but the countably infinite union

$$\mathbb{Z} = \bigcup_{a\in\mathbb{Z}} \{a\}$$

is infinite but not all of $\mathbb{A}^1_{\mathbb{C}}$, and is thus not a subvariety. ■

2.1 The Zariski Topology

From now on, we assume that k is algebraically closed unless specified otherwise.

Recall that a topology on a set X is a set $T \subseteq \mathcal{P}(X)$ of *open sets* such that

(T1) X is open and $\varnothing$ is open;

(T2) The arbitrary union of open sets is open;

(T3) The finite intersection of open sets is open.

The complement of an open set is called *closed*.

For our purposes, it will be helpful to characterise topologies in terms of closed sets instead. By De Morgan's laws, a topology on X is equivalently a set $T' \subseteq \mathcal{P}(X)$ of closed sets such that

(T1) X is closed and $\varnothing$ is closed;

(T2) The arbitrary intersection of closed sets is closed;

(T3) The finite union of closed sets is closed.

Now, we have seen that $\mathbb{A}^n$ and $\varnothing$ are both subvarieties, and that the arbitrary intersection and finite unions (by induction on binary unions) of subvarieties are subvarieties.

So, the collection of subvarieties of $\mathbb{A}^n$ defines a topology of closed sets on $\mathbb{A}^n$ called the *Zariski topology.*

Compared to the standard topology, non-empty Zariski-open sets are very "large". While the standard topology has a basis consisting of open balls of arbitrarily small radius, every non-empty Zariski-open set is unbounded in the standard topology, and in fact dense in both the Zariski and standard topology. Furthermore, any two non-empty Zariski-open subsets have non-empty intersection, so the Zariski topology is strongly non-Hausdorff.

Note that this definition is also satisfied by affine algebraic sets when k is not algebraically closed, but we will generally be interested in the case of topologies of subvarieties.

Lemma 2.4. *Any subset of* $\mathbb{R}^n$ *or* $\mathbb{C}^n$ *that is closed in the Zariski topology is also closed in the standard topology.*

Proof. Let S be closed in the Zariski topology, so $S = \mathbb{V}\big(\{f_i\}_{i\in I}\big) = \bigcap_{i\in I} \mathbb{V}(f_i)$. Since polynomials are continuous with respect to the standard topology and $\{0\}$ is closed in the standard topology, $\mathbb{V}(f_i) = f_i^{-1}\big[\{0\}\big]$ is also closed, and hence the intersection S is also closed. ■

Recall that if X is a topological space and $Y \subseteq X$ is a subset, then Y is naturally a topological space under the *subspace topology*, where the open and closed sets of Y are the open and closed sets of X intersected with Y.

In particular, if Y is a subvariety of $\mathbb{A}^n$, then the Zariski-closed subsets of Y are subvarieties of $\mathbb{A}^n$ intersected with Y. Since intersections of subvarieties are subvarieties, the closed subsets of a subvariety Y are precisely the subvarieties of $\mathbb{A}^n$ that are contained in Y.

If Y is a subvariety of $\mathbb{A}^n$, then a *subvariety of* Y is a Zariski-closed subset of Y, or equivalently, a subvariety of $\mathbb{A}^n$ that is contained in Y.

Theorem 2.5. *Subvarieties are compact in the Zariski topology.*

2.2 Regular Maps

A map $f : \mathbb{A}^n \to \mathbb{A}^m$ is *regular* or is a *morphism of affine space* if every component is a polynomial. That is, there exist $f_1, \ldots, f_m \in k[x_1, \ldots, x_n]$ such that

$$f(x_1, \ldots, x_n) = \big(f_1(x_1, \ldots, x_n), \ldots, f_m(x_1, \ldots, x_n)\big)$$

Example. The projection map $f : \mathbb{A}^2_{x,y} \to \mathbb{A}^2_{x,y}$ defined by $(x,y) \mapsto x$ is a regular map, since the only component x is a polynomial. △

Example. The map $h : \mathbb{A}^1_t \to \mathbb{A}^2_{x,y}$ defined by $t \mapsto (t^2, t^3)$ is a regular map, since the two components t^2 and t^3 are polynomials. △

This definition naturally extends to subvarieties. If $V \subseteq \mathbb{A}^n$ and $W \subseteq \mathbb{A}^n$ are subvarieties, then a map of sets $f : V \to W$ is *regular* or is a *morphism of subvarieties* if it can be expressed as the restriction of a regular map $\mathbb{A}^n \to \mathbb{A}^m$.

Note that the extension $\mathbb{A}^n \to \mathbb{A}^m$ is not necessarily unique.

Example. Let $V = \mathbb{V}(y - x^2) \subseteq \mathbb{A}^2$ and $W = \mathbb{A}^1$. The map $f : V \to W$ defined by $(x,y) \mapsto x$ is a morphism, since it is the restriction of the regular map $\mathbb{A}^2_{x,y} \to \mathbb{A}^1_t : (x,y) \mapsto x$.

It is also the restriction of the map $(x,y) \mapsto x + y - x^2$, since $y - x^2 = 0$ on V. △

Lemma 2.6. *Let* $F : \mathbb{A}^n_{x_1,\ldots,x_n} \to \mathbb{A}^m_{y_1,\ldots,y_m}$ *be regular. Then, for any* $g \in k[y_1, \ldots, y_m]$,

(i) $g \circ F \in k[x_1, \ldots, x_n]$ *is a polynomial;*

(ii) $F^{-1}\big[\mathbb{V}(g)\big] = \mathbb{V}(g \circ F)$*;*

(iii) $F^{-1}\big[\mathbb{V}\big(\{g_i\}_{i\in I}\big)\big] = \mathbb{V}\big(\{g_i \circ F\}_{i\in I}\big)$.

Proof.

(*i*) Since F is regular, its components are polynomials $F_1,\dots,F_m \in k[x_1,\dots,x_n]$. Then, the composition is defined by the substitution:

$$g \circ F = g\big(F_1(x_1,\dots,x_n),\dots,F_m(x_1,\dots,x_n)\big)$$

But a polynomial combination of polynomials is again a polynomial, so $g \circ F \in k[x_1,\dots,x_n]$.

(*ii*) By definition, $x \in F^{-1}\big[\mathbb{V}(g)\big]$ if and only if $F(x) \in \mathbb{V}(g)$ if and only if $g\big(F(x)\big)$, or equivalently, $x \in \mathbb{V}(g \circ F)$.

(*iii*) Similar to the previous, $x \in F^{-1}\big[\mathbb{V}\big(\{g_i\}_{i\in I}\big)\big]$ if and only if $F(x) \in \mathbb{V}\big(\{g_i\}_{i\in I}\big)$ if and only if $g_i\big(F(x)\big)$ for all $i \in I$, or equivalently, $x \in \mathbb{V}\big(\{g_i \circ F\}_{i\in I}\big)$.

■

Corollary 2.6.1. *The regular preimage of a subvariety of $\mathbb{A}^m$ is a subvariety of $\mathbb{A}^n$.*

Lemma 2.7. *Let $X \subseteq \mathbb{A}^n$ and $Y \subseteq \mathbb{A}^m$ be subvarieties, and $F : X \to Y$ be regular. Then, for any subvariety $W \subseteq Y$, $F^{-1}[W]$ is also a subvariety of X.*

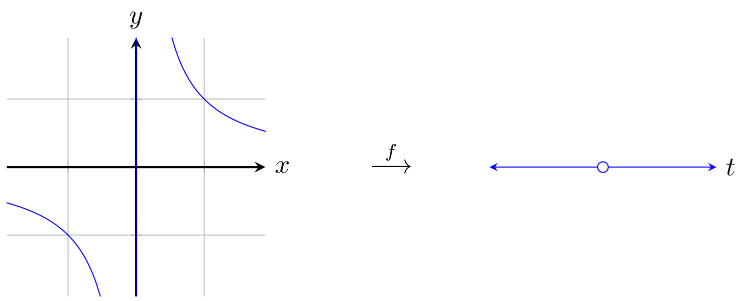
Proof. Since W is a subvariety of Y, it is given by $W = \mathbb{V}\big(\{f_i\}_{i\in I}\big)$. Let $F : \mathbb{A}^n \to \mathbb{A}^m$ be a regular map extending $F : X \to Y$. Then,

$$\begin{aligned} F^{-1}[W] &= \big\{x \in X : F(x) \in \mathbb{V}\big(\{f_i\}_{i\in I}\big)\big\} \\ &= \big\{x \in X : x \in \mathbb{V}\big(\{f_i \circ F\}_{i\in I}\big)\big\} \\ &= X \cap \mathbb{V}\big(\{f_i \circ F\}_{i\in I}\big) \end{aligned}$$

so $F^{-1}[W]$ is a subvariety of X. ■

Note, however, that the regular direct image of a subvariety is not necessarily a subvariety. That is, a regular map need not be a closed map.

Example. Let $V = \mathbb{V}(xy-1)$ be a subvariety of $\mathbb{A}^2_{x,y}$, and let $f : \mathbb{A}^2_{x,y} \to \mathbb{A}^1_t$ be the regular map $(x,y) \mapsto x$. Then, $f(V) = \mathbb{A}^1 \setminus \{0\}$, which is not a subvariety.



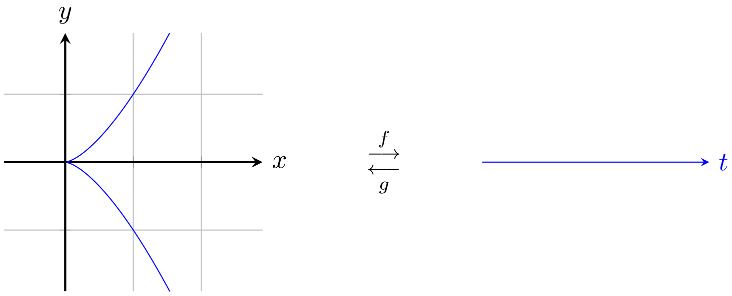
△

A regular map $V \to W$ is furthermore an *isomorphism* (*of subvarieties*) if it has a regular inverse. Two subvarieties are *isomorphic* if there exists an isomorphism between them, and we denote this relation as usual by $V \cong W$.

Example. Consider the subvarieties $V = \mathbb{V}(y - x^2) \subseteq \mathbb{A}^2_{x,y}$ and $W = \mathbb{A}^1_t$. Then, the regular map $f : \mathbb{A}^2_{x,y} \to \mathbb{A}^1_t : (x,y) \mapsto x$ has inverse given by the restriction of the regular map $g : \mathbb{A}^1_t \to \mathbb{A}^2_{x,y} : t \mapsto (t,t^2)$.

△

Example. Consider the subvarieties $V = \mathbb{V}(y^2 - x^3) \subseteq \mathbb{A}^2_{x,y}$ and $W = \mathbb{A}^1_t$. The regular map $g : \mathbb{A}^1_t \to \mathbb{A}^2_{x,y}$ defined by $t \mapsto (t^2,t^3)$ is a bijection, and is moreover a homeomorphism in the Zariski topology, but is not an isomorphism of subvarieties.



Since $x = t^2$ and $y = t^3$, the inverse is given by either $\sqrt{x}$, $\sqrt[3]{y}$, or $\frac{y}{x}$; none of which are polynomial. △

2.3 Irreducibilty

A topological space X is *reducible* if it is the union of two distinct closed proper subsets, and is *irreducible* otherwise.

Example. The interval $[0,1]$ is reducible in the standard topology, since $[0,1] = [0,\frac{1}{2}],[\frac{1}{2},1]$. △

Example. Any one-point space is irreducible, since any proper subset is necessarily empty. △

By convention, the empty set is considered irreducible.

Lemma 2.8. *$\mathbb{A}^1_k$ is irreducible in the Zariski topology for any infinite field k.*

Proof. Any proper closed subset of $\mathbb{A}^1_k$ is either finite or empty, but $\mathbb{A}^1_k$ is infinite, and thus cannot be the union of two such subsets. ■

Theorem 2.9. *Every subvariety V of $\mathbb{A}^n$ can be uniquely expressed as the union of finitely many irreducible subvarieties.*

The irreducible subvarieties in this decomposition are called the *irreducible components* of V.

Lemma 2.10. *If V is a subvariety and $W \subseteq V$ is an irreducible subvariety, then W is contained in one of the irreducible components of V.*

Proof. If $V = V_1 \cup V_2$ are proper closed subsets and $W \subseteq V$ is closed, then $W = (W \cap V_1) \cup (V \cap V_2)$ are both closed. So W is irreducible only if it is contained entirely within V_1 or V_2. ■

If k is not algebraically closed, then qualitatively different polynomials can have the same vanishing loci. For instance, over $\mathbb{R}$ as *sets*, $\mathbb{V}(x^2+y^2) = \mathbb{V}(x,y) = \{(0,0)\} \subseteq \mathbb{A}^2_{\mathbb{R}}$, but we would really like to be able to distinguish these as *subvarieties.*

The problem is that, over $\mathbb{C}$, $\mathbb{V}(x^2+y^2) \subseteq \mathbb{A}^2_{\mathbb{C}}$ is reducible as $\mathbb{V}(x^2+y^2) = \mathbb{V}(x+iy) \cup \mathbb{V}(x-iy)$. The regular maps $\mathbb{V}(x^2+y^2) \to \mathbb{A}^1$ defined by $(x,y) \mapsto 0$ and $(x,y) \mapsto x$ agree as maps of sets over $\mathbb{R}$, but not over $\mathbb{C}$. Even over $\mathbb{R}$, we say that these two maps are distinct *as morphisms.*

The point is that, over an algebraically closed field, the set-theoretic picture faithfully captures the algebro-geometric situation, while over $\mathbb{R}$, it is not enough.

Lemma 2.11. *The continuous image of an irreducible space is irreducible.*

Proof. Let X is irreducible and $f : X \to Y$ is continuous and surjective. Suppose that $Y = Y_1 \cup Y_2$ are both closed. Then, since f is continuous, $X = f^{-1}[Y_1] \cup F^{-1}[Y_2]$ are both closed. Since X is irreducible, (at least) one of these must be equal to X, say $f^{-1}[Y_1]$. But then, by the surjectivity of f, $Y = f(X) = Y_1$, and Y is irreducible. ■

Theorem 2.12. *Let X be a topological space, and $V \subseteq X$ be a subspace. Then, V is irreducible if and only if $\overline{V}$ is irreducible.*

Let X be a topological space and $V \subseteq X$ be a subspace. Recall that V is *dense* in X if $\overline{V} = X$.

A map $f : X \to Y$ of topological spaces is *dominant* if $f(X)$ is dense in Y.

Example. Consider the map $f : \mathbb{A}^2 \to \mathbb{A}^2$ defined by $(x,y) \mapsto (xy,y)$. Then, $f(\mathbb{A}^2) = \big(\mathbb{A}^2 \setminus \{x\text{-axis}\}\big) \cup \big\{(0,0)\big\}$. This is neither open nor closed, but is dense in $\mathbb{A}^2$, so f is dominant. △

2.4 Dimension

Consider the subvariety of $\mathbb{A}^3$ defined by $\mathbb{V}(x^2 + y^2 + z^2 - 1)$. It sees reasonable that the "dimension" of this subvariety should be 2, since it can be thought of as a complex 2-sphere.

What about the subvariety $V = \mathbb{V}(xy,xz) = \mathbb{V}(x) \cup \mathbb{V}(y,z) = \{yz\text{-plane}\} \cup \{x\text{-axis}\}$ of $\mathbb{A}^3$? This variety has two components: the yz-plane, which has dimension 2; and the x-axis which has dimension one. We adopt the convention that the subvariety V should have dimension two.

The *Krull dimension* $\dim(V)$ of a subvariety $V \subseteq \mathbb{A}^n$ is the length d of the longest possible chain $V_0 \subset V_1 \subset \cdots V_{d-1} \subset V_d$ of non-empty irreducible subvarieties of V.

Example. $\mathbb{A}^1$ is 1-dimensional, since its only proper irreducible subvarieties of $\mathbb{A}^1$ are singletons, $\{\text{pt}\} \subseteq \{\text{line}\}$. △

Theorem 2.13. *$\mathbb{A}^n$ is n-dimensional.*

Note that if $V_0 \subset \cdots V_d$ is such a maximal chain, then necessarily $\dim(V_k) = k$ and $V_0 = \{\text{pt}\}$. If V is irreducible, then we also have $V_d = V$.

Lemma 2.14. *The dimension of a subvariety of $\mathbb{A}^n$ is the maximum dimension of its irreducible components.*

Proof. Let $V = V_1 \cup \cdots \cup V_n$ all be irreducible subvarieties with no containments between any of the V_i. By Lemma 2.10, every irreducible subvariety of V lies in one of the V_i, so any chain of irreducible subvarieties of V is also a chain in that V_i. So the dimension of V is at most that of the maximum dimension of the V_i. Conversely, any chain in a V_i is also a chain in V, so the dimension of V is also at least that of the maximum dimension of the V_i. ■

A subvariety is *equidimensional* if all of its irreducible components have the same dimension.

Example. The subvariety $\mathbb{V}(xy,xz) = \mathbb{V}(x) \cup \mathbb{V}(y,z)$ is not equidimensional since $\mathbb{V}(x)$ is 2-dimensional, while $\mathbb{V}(y,z)$ is 1-dimensional. △

Lemma 2.15. *If W is a subvariety of V, then* $\dim(W) \leq \dim(V)$.

Proof. Any chain in W is also a chain in V. ■

Lemma 2.16. *If $f : X \to Y$ is a surjective regular map of subvarieties, then* $\dim(X) \geq \dim(Y)$.

That is, the dimension of the image is at most the dimension of the source: there are no space-filling curves in algebraic geometry. In fact, we can weaken surjectivity to dominance, and the result still holds:

Theorem 2.17. *If $f : X \to Y$ is a dominant regular map of subvarieties, then* $\dim(X) \geq \dim(Y)$.

3 Algebraic Foundations

A $\mathbb{C}$-*algebra* is a commutative ring that contains $\mathbb{C}$ as a subring.

Example. Any polynomial ring $R = \mathbb{C}[x_1, \dots ,x_n]$ over $\mathbb{C}$ is a $\mathbb{C}$ algebra since the subspace of contant polynomials in R is isomorphic to $\mathbb{C}$. △

Every $\mathbb{C}$-algebra R is naturally a $\mathbb{C}$-vector space, where the addition of vectors is defined by the addition in R and the multiplication of a scalar in $\mathbb{C}$ by a vector in R is defined by the multiplication in R.

We can define concepts for $\mathbb{C}$-algebras that are analogous to that of rings and ideals:

- The $\mathbb{C}$-subalgebra *generated by* a subset S of a $\mathbb{C}$-algebra R is the set

$$\langle S\rangle = \bigcap_{\substack{S\subseteq A\subseteq R \\ A \text{ is a } \mathbb{C}\text{-algebra}}} A$$

  That is, the $\mathbb{C}$-algebra $\langle S\rangle$ is the smallest $\mathbb{C}$-algebra containing S, or equivalently, the collection of all finite *polynomial* combinations of elements of S with coefficients in R.

- A $\mathbb{C}$-algebra is *finitely generated* if there is a finite set S such that $A = \langle S\rangle$.

  For instance, the polynomial ring $\mathbb{C}[x,y]$ is finitely generated as a $\mathbb{C}$-algebra by the elements x and y.

- A ring homomorphism $\phi : R \to S$ between $\mathbb{C}$-algebras R and S is a $\mathbb{C}$-*algebra homomorphism* if it is additionally $\mathbb{C}$-linear.

Example. The complex conjugate map $z \mapsto \bar{z}$ is a ring homomorphism $\mathbb{C} \to \mathbb{C}$, but is not a $\mathbb{C}$-algebra homomorphism since it is not $\mathbb{C}$-linear. △

If R is a $\mathbb{C}$-algebra and $I \subseteq R$ is an ideal, then R/I is a $\mathbb{C}$-algebra and the quotient map $R \to R/I$ is a $\mathbb{C}$-algebra homomorphism.

Theorem (Universal Property of Polynomial Rings)**.** *Suppose R is a $\mathbb{C}$-algebra and $a_1, \dots ,a_n \in R$. Then, there exists a unique $\mathbb{C}$-algebra homomorphism $\phi : \mathbb{C}[x_1, \dots ,x_n] \to R$ such that $\phi(x_i) = a_i$.*

Example. Let $R = \mathbb{C}[t]$, and pick $t^2,t^3 \in \mathbb{C}[t]$. Then, there is a unique $\mathbb{C}$-algebra homomorphism $\phi : \mathbb{C}[x,y] \to \mathbb{C}[t]$ such that $\phi(x) = t^2$ and $\phi(y) = t^3$. △

Lemma 3.1. *Every finitely generated $\mathbb{C}$-algebra R is the quotient of a polynomial ring.*

Proof. Pick generators $a_1, \dots ,a_k$ of R. Then, by the universal property of polynomial rings, there is a unique $\mathbb{C}$-algebra homomorphism $\phi : \mathbb{C}[x_1, \dots ,x_k] \to R$ with $\phi(x_i) = a_i$.

Since the a_i generate R, ϕ is surjective, so by the first isomorphism theorem,

$$R = \operatorname{im}\phi \cong \mathbb{C}[x_1, \dots ,x_k]/\ker(\phi)$$

■

3.1 Hilbert's Basis Theorem

Although the definition of an affine subvariety allows for arbitrarily many polynomials in the vanishing locus, it turns out that every affine subvariety can be expressed as the vanishing locus of only finitely many polynomials. This follows from the *Noetherian* property of polynomials rings.

A ring is *Noetherian* if any of the following equivalent conditions hold:

- Every strictly ascending chain of ideals
$$I_0 \subset I_1 \subset \cdots$$
is finite.
- Every weakly ascending chain of ideals stabilises. That is, for every chain of ideals
$$I_0 \subseteq I_1 \subseteq \cdots$$
there exists $n > 0$ such that
$$I_n = I_{n+1} = I_{n+2} = \cdots$$
- Every ideal is finitely generated.

Lemma 3.2. *Every field is Noetherian.*

Lemma 3.3. *Let R be Noetherian and $I \subseteq R$ be an ideal. Then, every generating set for I contains a finite generating subset.*

Theorem (Hilbert's Basis Theorem)**.** *If R is Noetherian, then $R[x]$ is Noetherian.*

Corollary 3.3.1. *If R is Noetherian, then $R[x_1, \dots, x_n]$ is Noetherian.*

Corollary 3.3.2. *$\mathbb{C}[x_1, \dots, x_n]$ is Noetherian.*

3.2 Hilbert's Nullstellensatz

The set of polynomials that define a subvariety is not unique. Suppose that f and g vanish on a subvariety V of $\mathbb{A}^n$, and let h be any polynomial in $k[x_1, \dots, x_n]$. Then, $f+g$ and hf also vanish on X. In particular, if $V = \mathbb{V}(S)$, then adding $f + g$ and hf for any polynomial h to S does not change its zero locus. In other words,

Thus, we always have $\mathbb{V}\big(\langle S\rangle\big) = \mathbb{V}(S)$, where $\langle S\rangle$ is the ideal of $k[x_1, \dots, x_n]$ generated by S (that is, the set of all linear combinations of elements in S).

Let V be a subvariety of $\mathbb{A}^n$. Then, the set $\mathbb{I}(V)$ of polynomials that vanish on V is an ideal of $k[x_1, \dots, x_n]$ called the *vanishing ideal* by the same reasoning as above.

$$\mathbb{I}(V) = \big\{f \in k[x_1, \dots, x_n] : \forall x \in V, f(x) = 0\big\}$$

Lemma 3.4. *For any subvariety V of $\mathbb{A}^n$, the vanishing ideal $\mathbb{I}(V)$ is a radical ideal of $k[x_1, \dots, x_n]$.*

Proof. Let $f \in k[x_1, \dots, x_n]$ be such that $f^n \in \mathbb{I}(V)$ for some $n > 0$, so $f^n(x) = 0$ for all $x \in V$. Since k is a field, it has no zero divisors, so $f^n(x) = f(x)^n = 0$ if and only if $f(x) = 0$ for all $x \in V$. So $f \in \mathbb{I}(V)$. ∎

Theorem 3.5. *Every subvariety V is the vanishing locus of finitely many polynomials.*

Proof. Since $V = \mathbb{V}\big(\mathbb{I}(V)\big)$ and $k[x_1, \dots, x_n]$ is Noetherian, $\mathbb{I}(V) = \langle f_1, \dots, f_k\rangle$ is finitely generated, and hence
$$V = \mathbb{V}\big(\mathbb{I}(V)\big) = \mathbb{V}\big(\langle f_1, \dots, f_k\rangle\big) = \mathbb{V}(f_1, \dots, f_k)$$
∎

Theorem 3.6. *Every subvariety V of $\mathbb{A}^n$ is the intersection of finitely many hypersurfaces.*

Proof. $V = \mathbb{V}\big(\{f_i\}_{i\in I}\big) = \mathbb{V}\big(\langle f_i\rangle_{i\in I}\big)$. By Noetherianness, this ideal is finitely generated, so $\mathbb{V}\big(\langle f_i\rangle_{i\in I}\big) = \mathbb{V}\big(\langle f_1,\dots,f_k\rangle\big) = \bigcap_{i=1}^k \mathbb{V}\big(\langle f_i\rangle\big)$ is a finite intersection of hypersurfaces. ∎

Theorem 3.7. *For any subvariety V,*

$$\mathbb{V}\big(\mathbb{I}(V)\big) = V$$

Proof. By definition, $V \subseteq \mathbb{V}\big(\mathbb{I}(V)\big)$. Conversely, since V is a subvariety, $V = \mathbb{V}\big(\{f_i\}_{i\in I}\big)$, and by the definition of a vanishing ideal, $f_i \in \mathbb{I}(V)$ for each $i \in I$. Now, for any $x \in \mathbb{V}\big(\mathbb{I}(V)\big)$, x vanishes for each $f \in \mathbb{I}(V)$, and in particular, for each f_i, so $x \in \mathbb{V}\big(\{f_i\}_{i\in I}\big) = V$. ∎

So $\mathbb{V}$ is a left inverse to $\mathbb{I}$. What about the other order?

Theorem (Hilbert's Nullstellensatz)**.** *For any ideal $I \subseteq k[x_1,\dots,x_n]$,*

$$\mathbb{I}\big(\mathbb{V}(I)\big) = \sqrt{I}$$

With the previous result, $\mathbb{V}$ and $\mathbb{I}$ are inverse maps when restricted to radical ideals. In this way, Hilbert's Nullstellensatz implies a bijection

$$\big\{\text{affine subvarieties of } \mathbb{A}^n\big\} \underset{\mathbb{V}}{\overset{\mathbb{I}}{\rightleftarrows}} \big\{\text{radical ideals of } k[x_1,\dots,x_n]\big\}$$

So every radical ideal is in fact a vanishing ideal.

If V is a subvariety of W, then the functions vanishing on W also vanish on V. So $\mathbb{I}(V) \subseteq \mathbb{I}(W)$, so this correspondence is order-reversing. More generally, $\mathbb{V}$ is also order-reversing on all ideals, not necessarily radical.

Moreover, this order-reversing correspondence implies that every maximal ideal in $k[x_1,\dots,x_n]$ is the ideal of functions vanishing at a single point $(a_1,\dots,a_n) \in \mathbb{A}^n$. In particular, every maximal ideal has the form $\mathfrak{m}_a = \langle x_1 - a_1,\dots,x_n - a_n\rangle$, and the corresponding subvariety is the singleton $\mathbb{V}(\mathfrak{m}_a) = \{a\} = \big\{(a_1,\dots,a_n)\big\} \subseteq \mathbb{A}^n$. That is, under the above correspondence, the set of maximal ideals of $k[x_1,\dots,x_n]$ is identified with the points of affine n-space $\mathbb{A}^n$.

$$\big\{\text{points of } \mathbb{A}^n\big\} \underset{\mathbb{V}}{\overset{\mathbb{I}}{\rightleftarrows}} \big\{\text{maximal ideals of } k[x_1,\dots,x_n]\big\}$$

Similarly, prime ideals are identified with irreducible subvarieties under this correspondence, and radical ideals correspond to all subvarieties of $\mathbb{A}^n$.

$$\big\{\text{irreducible affine subvarieties of } \mathbb{A}^n\big\} \underset{\mathbb{V}}{\overset{\mathbb{I}}{\rightleftarrows}} \big\{\text{prime ideals of } k[x_1,\dots,x_n]\big\}$$

Lemma 3.8. *For any subset $S \subseteq \mathbb{A}^n$,*

$$\mathbb{V}\big(\mathbb{I}(S)\big) = \overline{S}$$

Proof. By definition, every polynomial in $\mathbb{I}(S)$ vanishes everywhere on S, so every point of S vanishes under $\mathbb{I}(S)$, i.e. $S \subseteq \mathbb{V}\big(\mathbb{I}(S)\big)$. Since $\mathbb{V}\big(\mathbb{I}(S)\big)$ is a closed set and $\overline{S}$ is the smallest closed set containing S, $\overline{S} \subseteq \mathbb{V}\big(\mathbb{I}(S)\big)$.

Conversely, let $T = \mathbb{V}\big(\{f_i\}_{i\in I}\big)$ be a closed set containing S, so $f_i \in \mathbb{I}(S)$ for all $i \in I$. Then, $\mathbb{V}\big(\mathbb{I}(S)\big) \subseteq \mathbb{V}\big(\{f_i\}_{i\in I}\big) = T$. Since $\mathbb{V}\big(\mathbb{I}(S)\big)$ is a subset of every closed set T containing S, it is a subset of the closure $\overline{S}$. ∎

Corollary 3.8.1. *If $V_1 \neq V_2$ are closed, then $\mathbb{I}(V_1) \neq \mathbb{I}(V_2)$.*

That is, subvarieties can be determined by their vanishing ideals.

Corollary 3.8.2. $\mathbb{I}(S_1) = \mathbb{I}(S_2)$ *if and only if* $\overline{S_1} = \overline{S_2}$.

Theorem 3.9. *Any strictly descending chain of subvarieties of* $\mathbb{A}^n_k$ *is finite.*

Proof. Let
$$V_1 \supset V_2 \supset V_3 \supset \cdots$$
a descending chain of subvarieties. Then,
$$\mathbb{I}(V_1) \subset \mathbb{I}(V_2) \subset \mathbb{I}(V_3) \subset \cdots$$
is an ascending chain of ideals. Since $k[x_1, \ldots, x_n]$ is Noetherian, this chain must be finite, and since $V_i = \mathbb{V}\big(\mathbb{I}(V_i)\big)$, the chain of subvarieties is finite. ■

Lemma 3.10. *Given* $\{f_i\} \subseteq k[x_1, \ldots, x_n]$,
$$\mathbb{I}\big(\mathbb{V}(\{f_i\}_{i\in I})\big) \supseteq \sqrt{\langle f_i \rangle}$$

Example. Let $k = \mathbb{R}$, and $I = \langle x^2 + 1\rangle \subseteq \mathbb{R}[x]$. I is a radical ideal, and $\mathbb{V}(x^2+1) = \varnothing$, so $\mathbb{I}\big(\mathbb{V}(x^2+1)\big) = \mathbb{I}(\varnothing) = \mathbb{R}[x] \supseteq \sqrt{I} = I$. △

4 The Coordinate Ring

We have seen a correspondence between the geometry of affine n-space $\mathbb{A}^n$ and various ideals of the $\mathbb{C}$-algebra $\mathbb{C}[x_1, \ldots, x_n]$. For a subvariety V, what $\mathbb{C}$-algebra describes the subvarieties of V?

Often, to understand an object, we instead study natural classes of functions defined on them. In topology, we study continuous functions on topological spaces; in differential geometry, we study smooth maps on smooth manifolds; and in complex geometry, we study holomorphic maps on complex manifolds. In algebraic geometry, the maps of choice are polynomials.

Let $V \subseteq \mathbb{A}^n$ be an affine subvariety. Given any polynomial $p \in \mathbb{C}[x_1, \ldots, x_n]$, the restriction $p\big|_V$ defines a function $V \to \mathbb{C}$. Under the usual pointwise addition and multiplication operations, the set of these functions naturally form a $\mathbb{C}$-algebra $\mathbb{C}[V]$ called the *coordinate ring* of V. In particular, the coordinate ring of the whole affine space $\mathbb{A}^n$ is $\mathbb{C}[\mathbb{A}^n] = \mathbb{C}[x_1, \ldots, x_n]$, as expected.

The elements of $\mathbb{C}[V]$ are restrictions of polynomials on $\mathbb{A}^n$, but we usually denote them by the original polynomials. This may be slightly confusing, as two ostensibly different polynomials may agree when restricted to V.

Example. Consider the variety $V = \mathbb{V}(y - x)$ of $\mathbb{A}^n$. Then, the polynomials $xy + 1$, $x^2 + 1$, and $y^2 + 1$ are all the same polynomial on V, since $y = x$ on V. △

Restriction defines a surjective ring homomorphism $\mathbb{C}[x_1, \ldots, x_n] \to \mathbb{C}[V]$. By definition, the kernel of this homomorphism is precisely the vanishing ideal $\mathbb{I}(V)$, so by the first isomorphism theorem,
$$\mathbb{C}[V] \cong \frac{\mathbb{C}[x_1, \ldots, x_n]}{\mathbb{I}(V)}$$

Which $\mathbb{C}$-algebras are coordinate rings?

Recall that a ring R is *reduced* if for all elements $x \in R$, whenever $x^n = 0$ for some $n > 0$, then $x = 0$. That is, 0 is the only nilpotent element in R.

Lemma 4.1. *An ideal* I *is radical if and only if* R/I *is reduced.*

Theorem 4.2. *R is a finitely generated reduced $\mathbb{C}$-algebra if and only if R is the coordinate ring of some affine subvariety V of affine space.*

Proof. As the quotient of a finitely generated $\mathbb{C}$-algebra $\mathbb{C}[x_1,\dots,x_n]$ by a radical ideal $\mathbb{I}(V)$, coordinate rings are finitely generated reduced $\mathbb{C}$-algebras.

Now, suppose R is a finitely generated reduced $\mathbb{C}$-algebra. Pick generators $a_1,\dots,a_n \in R$. Then, there exists a unique $\mathbb{C}$-algebra homomorphism $\Phi : \mathbb{C}[x_1,\dots,x_n] \to R$ with $\Phi(x_i) = a_i$. Because the a_i generate R, Φ is surjective, so by the first isomorphism theorem,

$$R \cong \frac{\mathbb{C}[x_1,\dots,x_n]}{\ker(\Phi)}$$

Since R is reduced, $\ker(\Phi)$ is radical and hence determines a subvariety $V = \mathbb{V}\big(\ker(\Phi)\big)$ of $\mathbb{A}^n$. So, $\ker(\Phi) = \mathbb{I}(V)$ and hence R is the coordinate ring $\mathbb{C}[V]$. ■

Specifically, the relations between the generators $a_1,\dots,a_n$ determine the ideal $\mathbb{I}(V)$. Selecting a different set of generators $b_1,\dots,b_m \in R$ would then yield a different subvariety W of $\mathbb{A}^m$ with $R \cong \mathbb{C}[W]$. We will soon define a notion of isomorphism between subvarieties that identify W and V.

Example. Let $R = \mathbb{C}[t]$ and consider the generators $x = t$ and $y = 1$. These generators satisfy the relation $y - 1 = 0$, so

$$R \cong \frac{\mathbb{C}[x,y]}{\langle y-1 \rangle}$$

This choice of generators corresponds to the subvariety $V = \mathbb{V}(y-1) \subseteq \mathbb{A}^2_{x,y}$.

If we instead choose the generators $a = t^2$, $b = 2t$, and $c = 1-t$, the relations are then $a - b - c^2 + 1 = 0$, so

$$R \cong \frac{\mathbb{C}[a,b,c]}{\langle a - b - c^2 + 1 \rangle}$$

This choice of generators corresponds to the subvariety $W = \mathbb{V}(a - b - c^2 + 1) \subseteq \mathbb{A}^3_{a,b,c}$. △

So far, we have been viewing subvarieties as certain subsets of affine n-space $\mathbb{A}^n$. However, as we have seen above, we cannot precisely recover a subvariety V from the coordinate ring $\mathbb{C}[V]$, and can only do so up to isomorphism.

The subvarieties V and W are all isomorphic and embed the same subvariety $\mathbb{A}^1$ into $\mathbb{A}^2$ and $\mathbb{A}^3$ respectively. So, we want a definition that captures the notion of a subvariety is an entity by itself that does not depend on an ambient embedding.

Before, we have seen a correspondence between certain geometric features of $\mathbb{A}^n$ and the polynomial ring $\mathbb{C}[x_1,\dots,x_n] = \mathbb{C}[\mathbb{A}^n]$. We will now establish an analogous correspondence between geometric features of an arbitrary affine subvariety V and its coordinate ring $\mathbb{C}[V]$.

Recall that, given an ideal I of R, the quotient map $\pi : R \to R/I$ induces a bijection

$$\begin{aligned} \{\text{ideals of } R/I\} &\xrightarrow{\cong} \{\text{ideals of } R \text{ containing } I\} \\ J &\mapsto \pi^{-1}[J] \end{aligned}$$

Since $\frac{R/I}{J} \cong \frac{R}{\pi^{-1}[J]}$, this bijection sends prime, maximal, and radical ideals to prime, maximal, and radical ideals, respectively.

$$\big\{\text{ideals of } \mathbb{C}[V]\big\} \cong \left\{\begin{matrix}\text{ideals of} \\ \mathbb{C}[x_1,\dots,x_n] \text{ containing } I\end{matrix}\right\}$$

$$\begin{Bmatrix}\text{radical}\\ \text{ideals of } \mathbb{C}[V]\end{Bmatrix} \cong \begin{Bmatrix}\text{radical ideals of}\\ \mathbb{C}[x_1,\ldots,x_n] \text{ containing } I\end{Bmatrix} \cong \begin{Bmatrix}\text{all}\\ \text{subvarieties } W \subseteq V\end{Bmatrix}$$
$$\begin{Bmatrix}\text{prime}\\ \text{ideals of } \mathbb{C}[V]\end{Bmatrix} \cong \begin{Bmatrix}\text{prime ideals of}\\ \mathbb{C}[x_1,\ldots,x_n] \text{ containing } I\end{Bmatrix} \cong \begin{Bmatrix}\text{irreducible}\\ \text{subvarieties } W \subseteq V\end{Bmatrix}$$
$$\begin{Bmatrix}\text{maximal}\\ \text{ideals of } \mathbb{C}[V]\end{Bmatrix} \cong \begin{Bmatrix}\text{maximal ideals of}\\ \mathbb{C}[x_1,\ldots,x_n] \text{ containing } I\end{Bmatrix} \cong \begin{Bmatrix}\text{points}\\ (a_1,\ldots,a_n) \in V\end{Bmatrix}$$

Thus, the Zariski topology on V, the points of V, and the subvarieties of V are all encoded in the $\mathbb{C}$-algebra structure of $\mathbb{C}[V]$. This gives us a way to think of a subvariety V independently from its original construction as a subset of $\mathbb{A}^n$.

An *affine variety* is a subvariety of $\mathbb{A}^n$ together with its Zariski topology and coordinate ring.

4.1 The Pullback Homomorphism

Just as each affine variety induces a unique $\mathbb{C}$-algebra as its coordinate ring, every morphism of affine varieties determines a unique $\mathbb{C}$-algebra homomorphism between their coordinate rings *in reverse direction.*

Given any morphism $F : V \to W$, there is a map of coordinate rings $\mathbb{C}[W] \to \mathbb{C}[V]$ defined by precomposition by F:

$$\begin{aligned}\mathbb{C}[W] &\to \mathbb{C}[V]\\ g &\mapsto g \circ F\end{aligned}$$

called the *pullback* of F, denoted by $F^\sharp$.

Lemma 4.3. *For any morphism $F : V \to W$, the pullback map $F^\sharp : \mathbb{C}[W] \to \mathbb{C}[V]$ is a $\mathbb{C}$-algebra homomorphism.*

Example. Consider the morphism F of affine varieties defined by

$$\begin{aligned}\mathbb{A}^3_{x,y,z} &\to \mathbb{A}^2_{u,v}\\ (x,y,z) &\mapsto (x^2y, x - z)\end{aligned}$$

The pullback $F^\sharp$ is then defined by

$$\begin{aligned}\mathbb{C}[u,v] &\to \mathbb{C}[x,y,z]\\ u &\mapsto x^2y\\ v &\mapsto x - z\end{aligned}$$

△

Lemma 4.4. *Given distinct points $p,q \in V$, there exists a polynomial $g \in \mathbb{C}[V]$ such that $g(p) \neq g(q)$.*

Proof. Since $V \subseteq \mathbb{A}^n_{x_1,\ldots,x_n}$, if $p \neq q$, then they must differ in some coordinate x_i. Then, the polynomial x_i will do. ■

Theorem 4.5. *Given morphisms $F,G : V \to W$, $F = G$ if and only if $F^\sharp = G^\sharp$.*

Proof. The forward direction is obvious. Conversely, suppose $F \neq G$, so there exists $x \in V$ such that $F(x) \neq G(x)$. By the previous lemma, there is a polynomial $g \in \mathbb{C}[W]$ such that

$$F^\sharp(g)(x) = g\big(F(x)\big) \neq g\big(G(x)\big) = G^\sharp(g)(x)$$

so $F^\sharp(g) \neq G^\sharp(g)$, and hence $F^\sharp \neq G^\sharp$. ■

4.2 The Equivalence of Algebra and Geometry

Lemma 4.6. *Given $F : V \to W$ and $G : W \to X$, $(G \circ F)^\sharp = F^\sharp \circ G^\sharp$.*

Theorem 4.7. *Every homomorphism $\sigma : S \to R$ of finitely generated reduced $\mathbb{C}$-algebras can be realised essentially uniquely as the pullback of a morphism $F : V \to W$ of affine varieties.*

That is, there exist affine varieties V and W with identifications $\mathbb{C}[V] \cong C$ and $\mathbb{C}[W] \cong S$ and a morphism $F : V \to W$ such that under these identifications, $F^\sharp = \sigma$:

$$\begin{tikzcd}
\mathbb{C}[V] \arrow[dr, leftrightarrow, "\cong"] & & \mathbb{C}[W] \arrow[ll, "F^\sharp"'] \arrow[dl, leftrightarrow, "\cong"'] \\
& S & R \arrow[l, "\sigma"']
\end{tikzcd}$$

Furthermore, the choices of V, W, and F are unique up to unique isomorphism, so if there exist V' and W' with identifications $\mathbb{C}[V'] \cong R$ and $\mathbb{C}[W] \cong S$, and a morphism $F' : V' \to W'$ such that under these identifications $(F')^\sharp = \sigma$, then there exist unique isomorphisms $\alpha : V \to V'$ and $\beta : W \to W'$ such that the following diagram commutes:

$$\begin{tikzcd}
\mathbb{C}[V] \arrow[dr, leftrightarrow, "\cong"] & & & \mathbb{C}[W] \arrow[lll, "F^\sharp"'] \arrow[dl, leftrightarrow, "\cong"'] \\
& S & R \arrow[l, "\sigma"'] & \\
\mathbb{C}[V'] \arrow[uu, dashed, "\alpha^\sharp"] \arrow[ur, leftrightarrow, "\cong"'] & & & \mathbb{C}[W'] \arrow[lll, "(F')^\sharp"] \arrow[ul, leftrightarrow, "\cong"] \arrow[uu, "\beta^\sharp"']
\end{tikzcd}$$

Corollary 4.7.1. *$V \cong W$ if and only if $\mathbb{C}[V] \cong \mathbb{C}[W]$.*

In summary, just as geometry determines algebra, algebra also determines geometry; every finitely generated reduced $\mathbb{C}$-algebra is equivalent to an affine variety V via $R \cong \mathbb{C}[V]$, and every homomorphism $\phi : S \to R$ of $\mathbb{C}$-algebras is equivalent to a morphism $F : V \to W$ of affine varieties by pullback.

5 The Spectrum of a Ring

We have seen how $\mathbb{C}[V]$ determines V up to isomorphism; by picking n generators for $\mathbb{C}[V]$, we obtain an embedding $V \hookrightarrow \mathbb{A}^n$. However, we can reconstruct V from $\mathbb{C}[V]$ "abstractly", not as a subset of $\mathbb{A}^n$.

The *maximal spectrum* of a commutative ring R is the set of maximal ideals of R:

$$\operatorname{maxSpec}(R) \coloneqq \{\mathfrak{m} \subset R : \mathfrak{m} \text{ is a maximal ideal}\}$$

So, by Hilbert's Nullstellensatz, there is a bijection

$$\begin{aligned} V &\xrightarrow{\cong} \operatorname{maxSpec}\big(\mathbb{C}[V]\big) \\ a &\mapsto \mathfrak{m}_x = \mathbb{I}\big(\{a\}\big) \end{aligned}$$

We can transport the Zariski topology on V to a topology on $\operatorname{maxSpec}\big(\mathbb{C}[V]\big)$ as follows. The points of a Zariski-closed set $W \subseteq V$ correspond to the set of maximal ideals in $\mathbb{C}[V]$ that contain $\mathbb{I}[W]$, so the closed sets of $\operatorname{maxSpec}\big(\mathbb{C}[V]\big)$ are sets of maximal ideals of $\mathbb{C}[V]$ containing some given ideal of $\mathbb{C}[V]$.

Given any commutative ring R and an ideal I of R, we define a notion of a vanishing locus on the maximal spectrum as:

$$\mathbb{V}^{\text{ms}}(I) \coloneqq \big\{\mathfrak{m} \in \operatorname{maxSpec}(R) : \mathfrak{m} \supseteq I\big\} \subseteq \operatorname{maxSpec}(R)$$

The Zariski topology on $\operatorname{maxSpec}(R)$ is the topology whose closed sets are precisely the sets of the form $\mathbb{V}^{\mathrm{ms}}(I)$ for I an ideal of R.

This gives us a way to define affine varieties without reference to any embeddings in affine space; for any finitely generated reduced $\mathbb{C}$-algebra R, $\operatorname{maxSpec}(R)$ is the corresponding affine variety.

Note, however, that the definition of a maximal spectrum and vanishing locus apply to any commutative ring, and not only finitely generated reduced $\mathbb{C}$-algebras.

Example. Let $R = \mathbb{Z}$. Then,
$$\operatorname{maxSpec}(R) = \big\{ \langle p \rangle : p \text{ is prime} \big\}$$
Since $\mathbb{Z}$ is a principle ideal domain, every ideal is of the form $I = \langle n \rangle$, so
$$\begin{aligned} \mathbb{V}^{\mathrm{ms}}(I) &= \big\{ \langle p \rangle : \langle m \rangle \subseteq \langle p \rangle \big\} \\ &= \big\{ \langle p \rangle : p \mid m \big\} \end{aligned}$$

△

We have seen that every morphism $V \to W$ induces a $\mathbb{C}$-algebra homomorphism $\mathbb{C}[W] \to \mathbb{C}[V]$ by pullback. Does every morphism also induce a morphism $\operatorname{maxSpec}\big(\mathbb{C}[V]\big) \to \operatorname{maxSpec}\big(\mathbb{C}[W]\big)$ of maximal spectra?

Lemma 5.1. *Let $F : V \to W$ be a morphism, and let $x \in V$. Then,*
$$(F^\sharp)^{-1}(\mathfrak{m}_x) = \mathfrak{m}_{F(x)}$$

Given a $\mathbb{C}$-algebra homomorphism $\sigma : \mathbb{C}[W] \to \mathbb{C}[V]$, we define the map of maximal spectra
$$\begin{aligned} \sigma^\flat : \operatorname{maxSpec}\big(\mathbb{C}[V]\big) &\to \operatorname{maxSpec}\big(\mathbb{C}[W]\big) \\ \mathfrak{m} &\mapsto \sigma^{-1}[\mathfrak{m}] \end{aligned}$$

Then, for any morphisms $F : V \to W$, the following diagram commutes:
$$\begin{tikzcd} \operatorname{maxSpec}\big(\mathbb{C}[V]\big) \arrow[r, "(F^\sharp)^\flat"] & \operatorname{maxSpec}\big(\mathbb{C}[W]\big) \\ V \arrow[u, "\cong"] \arrow[r, "F"'] & W \arrow[u, "\cong"'] \end{tikzcd}$$

So, morphisms of affine varieties are recoverable purely algebraically from maximal spectra of finitely generated reduced $\mathbb{C}$-algebras.

It seems that affine varieties are well-described by maximal spectra, and we may think to generalise this theory to maximal spectra of arbitrary commutative rings. Unfortunately, while the maximal spectrum construction still yields a topological space, we run into trouble when constructing maps between maximal spectra.

For instance, given a ring homomorphism $\sigma : R \to S$, we would like
$$\begin{aligned} \sigma^\flat : \operatorname{maxSpec}(S) &\to \operatorname{maxSpec}(R) \\ \mathfrak{m} &\mapsto \sigma^{-1}[\mathfrak{m}] \end{aligned}$$

to be a well-defined continuous map of topological spaces. However, the preimage of a maximal ideal under an arbitrary ring homomorphism is not necessarily a maximal ideal.

Example. Let $\sigma : \mathbb{Z} \hookrightarrow \mathbb{Q}$ be the inclusion map. As a field, the trivial ideal is maximal, but the preimage of the trivial ideal in $\mathbb{Q}$ is the trivial ideal in $\mathbb{Z}$, which is not maximal. △

Theorem 5.2. *The preimage of a prime ideal under a ring homomorphism is prime.*

Proof. Let $\phi : R \to S$ be a ring homomorphism, I be an ideal of S, and $J = \phi^{-1}[I]$. Suppose that $J = R$. Then, $1_R \in J$, so $\phi(1_R) = 1_S \in I$, so $I = S$, contradicting that I is prime. So J is a proper ideal.

Now, suppose $ab \in J$, so $\phi(ab) = \phi(a)\phi(b) \in I$. Since I is prime, $\phi(a) \in I$ or $\phi(b) \in I$, so $a \in J$ or $b \in J$. So J is prime. ■

The *spectrum* of a commutative ring is the set of *prime* ideals of R:

$$\operatorname{Spec}(R) \coloneqq \{\mathfrak{p} \subset R : \mathfrak{p} \text{ is a prime ideal}\}$$

Again, we define a notion of a vanishing locus on the spectrum as:

$$\mathbb{V}^{\mathrm{s}}(I) \coloneqq \big\{\mathfrak{p} \in \operatorname{Spec}(R) : \mathfrak{p} \supseteq I\big\} \subseteq \operatorname{Spec}(R)$$

and the Zariski topology on $\operatorname{Spec}(R)$ is again the topology whose closed sets are precisely the sets of the form $\mathbb{V}^{\mathrm{s}}(I)$ for I an ideal of R.

The spectrum of a ring, equipped with its Zariski topology, is what Grothendieck called an *affine scheme*.

While the maximal spectrum $\operatorname{maxSpec}\big(\mathbb{C}[V]\big)$ of a coordinate ring $\mathbb{C}[V]$ is canonically isomorphic to V, the spectrum $\operatorname{Spec}\big(\mathbb{C}[V]\big)$ contains more information and is canonically isomorphic to the set of irreducible subvarieties of V.

We write V^{sch} to abbreviate $\operatorname{Spec}\big(\mathbb{C}[V]\big)$.

Example. TODO △

6 Morphisms of Affine Schemes

Since primeness of ideals is preserved under ring homomorphism preimages, given a ring homomorphism $\sigma : S \to R$, we can again define a map of spectra:

$$\begin{aligned} \sigma^\flat : \operatorname{Spec}(R) &\to \operatorname{Spec}(S) \\ \mathfrak{p} &\mapsto \sigma^{-1}[\mathfrak{p}] \end{aligned}$$

Lemma 6.1. *$\sigma^\flat$ is continuous with respect to the Zariski topology.*

A *morphism of affine schemes* is the data of a ring homomorphism $\sigma : S \to R$ inducing a map $\sigma^\flat : \operatorname{Spec}(R) \to \operatorname{Spec}(S)$

An *affine scheme over* $\mathbb{C}$ is a spectrum $\operatorname{Spec}(R)$ of a not necessarily reduced or finitely generated $\mathbb{C}$-algebra R.

7 Projective Varieties

Let k be any field. Then, *n-dimensional projective space over k*, denoted $\mathbb{P}^n_k$ is the set of 1-dimensional subspaces of k^{n+1}. We will write $\mathbb{P}^n$ for n-dimensional complex projective space $\mathbb{P}^n_{\mathbb{C}}$.

Projective n-space can also be interpreted as the quotient

$$\mathbb{P}^n = \frac{\mathbb{C}^{n+1} \setminus \big\{(0,\dots,0)\big\}}{\sim}$$

where $\sim$ identifies two points that lie on the same line through the origin. That is, $(x_0, \ldots, x_n) \sim (y_0, \ldots, y_n)$ if and only if there exists a non-zero scalar $\lambda \in \mathbb{C}$ such that $(y_0, \ldots, y_n) = \lambda(x_0, \ldots, x_n)$.

A point in this space can then an equivalence class

$$\big[(z_0, \ldots, z_n)\big] = \big\{(\lambda z_0, \ldots, \lambda z_n) : \lambda \in \mathbb{C}\big\}$$

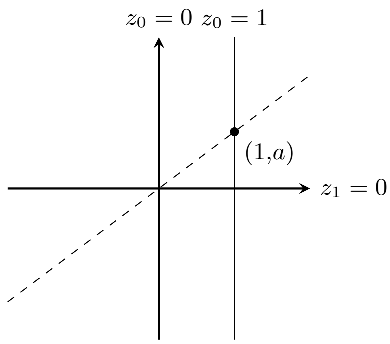
where at least one of the coordinates $z_0, \ldots, z_n$ must be non-zero. We denote a representative of the equivalence class of a point $(z_0, \ldots, z_n)$ by $[z_0 : \cdots : z_n]$, called a *homogeneous coordinate*. This notation emphasises that homogeneous coordinates are really just ratios of coordinates, and are defined only up to non-zero scaling: $[z_0 : \cdots : z_n] = [\lambda z_0 : \cdots : \lambda z_n]$ for any non-zero $\lambda \in \mathbb{C}$.

$$\mathbb{P}^n = \big\{[z_0 : \ldots : z_n] : \exists i, z_i \neq 0\big\}$$

Example. $[1 : 2] = [\frac{1}{2} : 1] = [i : 2i] \in \mathbb{P}^1$. These all represent the line $\big\{(z_0, z_1) : z_1 = 2z_0\big\} \subseteq \mathbb{C}^2$. △

0-dimensional projective space $\mathbb{P}^0$ is the set of all complex lines through the origin in $\mathbb{C}^1$, of which there is only $\mathbb{C}^1$ itself, so $\mathbb{P}^0 = \{\mathbb{C}^1\}$ is a singleton.

1-dimensional projective space $\mathbb{P}^1$ is the set of all complex lines through the origin in $\mathbb{C}^2$. By fixing a reference line – any complex line not through the origin, say $z_0 = 1$ – we can choose a representative for almost every point as the unique point on the reference line where the line through the origin intersects the reference line. Only one point in $\mathbb{P}^1$ will fail to have a representative under this scheme, namely the unique line through the origin parallel to the reference line, called the *point at infinity*.



Each line L_a of slope a can be represented by homogeneous coordinate:

$$\begin{aligned} L_a &= \big\{(z_0, az_0)\big\} \\ &= [z_0 : az_0] \\ &= [1 : a] \end{aligned}$$

while the vertical line $z_0 = 0$ has coordinate:

$$\begin{aligned} L_\infty &= \{z_0 = 0\} \\ &= [0 : z_0] \\ &= [0 : 1] \end{aligned}$$

So, we have a bijection

$$\begin{aligned} \mathbb{P}^1 \setminus \big\{[0 : 1]\big\} &\cong \{\text{coordinates with } z_0 = 1\} \\ L_a = [z_0, az_0] &\mapsto [1, a] \end{aligned}$$

and by discarding the first coordinate, we have a further isomorphism (as affine varieties):

$$\{\text{coordinates with } z_0 = 1\} \cong \mathbb{A}^1_{\mathbb{C}}$$

$$[1,a] \mapsto a$$

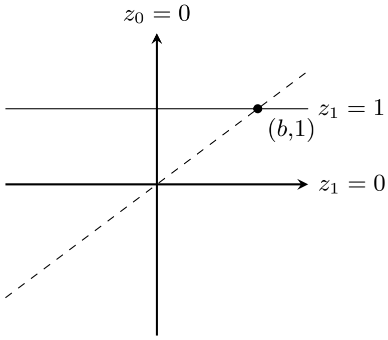
or directly,

$$\mathbb{P}^1 \setminus \{[0:1]\} \cong \mathbb{A}^1_{\mathbb{C}}$$
$$[z_0,z_1] \mapsto \tfrac{z_1}{z_0}$$

Adding in the point at infinity, this identifies $\mathbb{P}^1$ with the Riemann sphere

$$\mathbb{P}^1 \cong \mathbb{C} \cup \{\infty\}$$
$$[z_0,z_1] \mapsto \begin{cases} \frac{z_1}{z_0} & z_0 \neq 0 \\ \infty & z_0 = 0 \end{cases}$$

We could have also fixed the horizontal line $z_1 = 1$:



This time, every line L_b with slope $\frac{1}{b}$ through the origin is of the form $[b:1]$, while the horizontal line $z_1 = 0$ has coordinate $[1:0]$.

In $\mathbb{P}^2$, we can similarly select a reference plane not passing through the origin, and identify lines through the origin with their intersection with the reference plane. The exceptions will be the lines through the origin parallel to the reference plane – that is, a copy of $\mathbb{P}^1$.

$$\mathbb{P}^2 \cong \mathbb{C}^2 \cup \mathbb{P}^1 = \mathbb{C}^2 \cup \mathbb{C} \cup \{\infty\}$$

For instance, if we have coordinates z_0,z_1,z_2 for $\mathbb{C}^3$, and select the reference plane $z_0 = 1$, the identification sends the homogeneous coordinate $[z_0 : z_1 : z_2] \in \mathbb{P}^2$ to $(\frac{z_1}{z_0},\frac{z_2}{z_0}) \in \mathbb{C}^2$ whenever $z_0 \neq 0$, and to $[z_1 : z_2] \in \mathbb{P}^1$ when $x_0 = 0$.

Generalising this to arbitrary dimensions,

$$\mathbb{P}^n \cong \mathbb{C}^n \cup \mathbb{P}^{n-1}$$
$$[z_0 : z_1 : \cdots : z_n] \mapsto \begin{cases} (\frac{z_1}{z_0},\ldots,\frac{z_n}{z_0}) & z_0 \neq 0 \\ [z_1 : \cdots : z_n] & z_0 = 0 \end{cases}$$

We define the set $\mathcal{U}_0$ to be the set of points for which z_0 is non-zero. Under the above mapping, U_0 is identified with the hyperplane $z_0 = 1$ in $\mathbb{C}^{n+1}$, which can be identified with $\mathbb{C}^n$:

$$[z_0 : z_1 : \cdots : z_n] = \left[1 : \frac{z_1}{z_0} : \cdots : \frac{z_n}{z_0}\right] \mapsto \left(1,\frac{z_1}{z_0},\ldots,\frac{z_n}{z_0}\right) \cong \left(\frac{z_1}{z_0},\ldots,\frac{z_n}{z_0}\right)$$

The remaining points for which $z_0 = 0$ are then the points at infinity; the lines through the origin in $\mathbb{C}^{n+1}$ parallel to the hyperplane $z_0 = 1$, isomorphic to $\mathbb{P}^{n-1}$.

The choice of z_0 in the above is arbitrary. We define the set $\mathcal{U}_i$ to be the subset of $\mathbb{P}^n$ of points with $z_i \neq 0$

$$\mathcal{U}_i = \big\{[z_0 : \cdots : z_n] \in \mathbb{P}^n : z_i \neq 0\big\} = \mathbb{P}^n \setminus \{\text{1-dimensional subspaces of } \mathbb{V}(z_0)\}$$

isomorphic to $\mathbb{A}^n$ by dividing by and discarding the ith component:

$$\begin{aligned} \psi_i : \mathcal{U}_i &\to \mathbb{A}^n \\ [z_0 : \cdots : z_n] &\mapsto \left(\frac{z_0}{z_i}, \ldots, \frac{\widehat{z_i}}{z_i}, \ldots \frac{z_n}{z_i}\right) \end{aligned}$$

This isomorphism is called the *ith affine chart on* $\mathbb{P}^n$.

Example. In $\mathbb{P}^1$, we have

$$\begin{aligned} \mathcal{U}_0 &= \big\{[z_0 : z_1] : z_0 \neq 0\big\} = \big\{[1 : a] : a \in \mathbb{C}\big\} \cong \mathbb{A}^1_a \\ \mathcal{U}_1 &= \big\{[z_0 : z_1] : z_1 \neq 0\big\} = \big\{[b : 1] : b \in \mathbb{C}\big\} \cong \mathbb{A}^1_b \end{aligned}$$

△

The collection of all the affine charts yields a cover of $\mathbb{P}^n$ by $n+1$ copies of $\mathbb{A}^n$:

$$\mathbb{P}^n = \bigcup_{i=0}^{n} \mathcal{U}_i$$

Since $\mathbb{P}^n$ is the quotient of $\mathbb{C}^{n+1} \setminus \{0\}$, this $\mathbb{P}^n$ inherits a standard quotient topology from $\mathbb{C}^{n+1}$. In this topology, the affine charts are open.

The open cover $\{\mathcal{U}_i\}$ of $\mathbb{P}^n$ defines an atlas making projective space a complex n-dimensional manifold; we can move between charts via the *transition functions*

$$\psi_j \circ \psi_i^{-1} : \psi_i(\mathcal{U}_i \cap \mathcal{U}_j) \to \psi_j(\mathcal{U}_i \cap \mathcal{U}_j)$$

and these are not only holomorphic, but in fact rational. For instance,

$$\psi_n \circ \psi_0^{-1}(a_1, \ldots, a_n) = \psi_n\big([1 : a_1 : \cdots : a_n]\big) = \left(\frac{1}{a_n}, \frac{a_1}{a_n}, \ldots, \frac{a_{n-1}}{a_n}\right)$$

Example. On the intersection $\mathcal{U}_0 \cap \mathcal{U}_1 \subseteq \mathbb{P}^1$, both components of a homogeneous coordinate are non-zero, and we can translate between the two as:

$$\mathbb{A}^1 \cong \mathcal{U}_0 \ni a \mapsto [1 : a] = [\tfrac{1}{a} : 1] = \tfrac{1}{a} \in \mathcal{U}_1 \cong \mathbb{A}^1$$

so $b = \frac{1}{a}$, and hence $\mathcal{U}_0 \cap \mathcal{U}_1 \cong \mathbb{A}^1 \setminus \{0\}$. △

7.1 Projective Varieties

An element $[z_0 : \cdots : z_n] \in \mathbb{P}^n$ has many representations, given by scaling every component by some $\lambda \neq 0$, so something like:

$$\big\{[z_0 : \cdots : z_n] : z_0 = 5\big\}$$

is *not* well-defined. However, if $z_i = 0$ then $\lambda z_i = 0$ for any λ, so this is well-defined, like the affine charts.

Similarly, given any non-constant polynomial $f \in \mathbb{C}[z_0, \ldots, z_n]$, the value of $f(z_0, \ldots, z_n)$ and $f(\lambda z_1, \ldots, \lambda z_n)$ may not agree, so f does not define a function $\mathbb{P}^n \to \mathbb{C}$ since its value depends on the choice of homogeneous coordinates.

A polynomial is *homogeneous* if all of its terms have the same total degree.

Example. The polynomial $x^2y + 2z^3 \in \mathbb{C}[x,y,z]$ is homegeneous of degree 3. △

Lemma 7.1. *If $f(z_0,\ldots,z_n)$ is homogeneous of degree d, then*

$$f(\lambda z_0,\ldots,\lambda z_n) = \lambda^d f(z_0,\ldots,z_n)$$

Example. If $f(x,y,z) = x^2y + z^3$, then

$$\begin{aligned} f(\lambda x,\lambda y,\lambda z) &= (\lambda x)^2(\lambda y) + (\lambda z)^3 \\ &= \lambda^3 x^2 y + \lambda z^3 \\ &= \lambda^3 f(x,y,z) \end{aligned}$$

△

Corollary 7.1.1. *If f is homogeneous and $p \in \mathbb{P}^n$, then either for all choices of representatives $[z_0 : \cdots : z_n]$ of p, $f(z_0,\ldots,z_n) = 0$; or for all choices of representatives $[z_0 : \cdots : z_n]$ of p, $f(z_0,\ldots,z_n) \neq 0$.*

That is, a homogeneous polynomials only vanish along lines through the origin; if a homegeneous polynomial vanishes at a point, it must vanish along the entire line through the origin containing that point. Thus, the set of zeros in $\mathbb{C}^{n+1}$ of a homogeneous polynomial is the union of complex lines through the origin. So, while a homogeneous polynomial in $n+1$ variables does not define a function on $\mathbb{P}^n$, it still makes sense to talk about its vanishing locus in $\mathbb{P}^n$.

If $f \in \mathbb{C}[z_0,\ldots,z_n]$ is a homogeneous polynomial, then we define its projective vanishing locus to be

$$\mathbb{V}(f) = \big\{[z_0 : \ldots : z_n] : f(z_0,\ldots,z_n) = 0\big\} \subseteq \mathbb{P}^n$$

A *subvariety of* $\mathbb{P}^n$ is the vanishing locus $\mathbb{V}\big(\{f_i\}_{i\in I}\big)$ of some collection $\{f_i\}_{i\in I} \subseteq \mathbb{C}[z_0,\ldots,z_n]$ of homogeneous polynomials in $n+1$ variables.

A *projective variety* is a closed subvariety of some $\mathbb{P}^n$.

The Zariski topology on $\mathbb{P}^n$ is then the topology in which the closed subsets are precisely the subvarieties of $\mathbb{P}^n$.

Example. Consider the projective subvariety $V = \mathbb{V}(x^2 + y^2 - z^2) \subseteq \mathbb{P}^2$. We can write it as the union of its coordinate charts:

$$V = (V \cap \mathcal{U}_x) \cup (V \cap \mathcal{U}_y) \cup (V \cap \mathcal{U}_z)$$

On the chart $\mathcal{U}_z$ defined by $z \neq 0$, the variety looks like a complex circle; identifying $\mathcal{U}_z$ with $\mathbb{C}^2$, the curve in $\mathcal{U}_z$ is defined by the vanishing locus of $x^2 + y^2 - 1$, while on the charts $\mathcal{U}_x$ and $\mathcal{U}_y$, the curves are given by $1 + y^2 - z^2 = 0$ and $x^2 + 1 - z^2 = 0$, respectively. △

As in the example, the intersection of any projective variety V with one of the affine charts of $\mathbb{P}^n$ is an affine variety. Specifically, if $\mathcal{U}_i$ is an open set of $\mathbb{P}^n$ where the component z_i is non-zero, isomorphic to $\mathbb{A}^n$, then setting the variable z_i to 1 in the defining polynomials for V yields a set of defining polynomials for $V \cap \mathcal{U}_i$. So, just as projective space is covered by affine charts, we can think of a projective variety as being covered by affine varieties.

Another way to visualise a projective variety in $\mathbb{P}^n$ is to imagine a cone-shaped variety in $\mathbb{C}^{n+1}$, but then to identify all points lying on the same line through the origin. The variety in $\mathbb{C}^{n+1}$ defined by a collection of homogeneous polynomials is then called the *affine cone* over the projective variety in $\mathbb{P}^n$ defined by the same homogeneous polynomials.

Given a not-necessarily homogeneous polynomial $f \in \mathbb{C}[z_0,\ldots,z_n]$, we say that f *vanishes* at a point $p \in \mathbb{P}^n$ or write $f(p) = 0$, if $f(z_0,\ldots,z_n)$ for all choices of representative $[z_0 : \cdots : z_n] = p$, or equivalently, if the line $L_p \subseteq \mathbb{C}^{n+1}$ corresponding to p is entirely contained within the affine vanishing locus $\mathbb{V}(f) \subseteq \mathbb{C}^{n+1}$.

Any polynomial $f \in \mathbb{C}[z_0, \ldots, z_n]$ can be expressed uniquely as a sum

$$f = f_0 + f_1 + \cdots + f_d$$

where f_i is homogeneous of degree i. The polynomial f_i is called the *ith homogeneous component* of f.

Theorem 7.2. *Let $f \in \mathbb{C}[z_0, \ldots, z_n]$ and $p \in \mathbb{P}^n$ such that $f(p) = 0$. Then, for each homogeneous component f_i of f, $f_i(p) = 0$*

Proof. Pick a representative $p = [p_0 : \cdots : p_n]$, so

$$\begin{aligned} 0 &= f(p_0, \ldots, p_n) \\ &= f_0(p_0, \ldots, p_n) + \cdots + f_d(p_0, \ldots, p_n) \end{aligned}$$

Since $f(p) = 0$, f also vanishes at $\lambda[p_0 : \cdots : p_n]$ for any non-zero $\lambda \in \mathbb{C}$:

$$\begin{aligned} 0 &= f(\lambda p_0, \ldots, \lambda p_n) \\ &= \lambda^0 f_0(p_0, \ldots, p_n) + \cdots + \lambda^d f_d(p_0, \ldots, p_n) \end{aligned}$$

This is a polynomial in λ that vanishes for all non-zero λ, and must therefore be the zero polynomial. So the coefficients $f_i(p_0, \ldots, p_n)$ must all also vanish. Since the choice of representative was arbitrary, $f_i(p) = 0$. ■

An ideal $I \subseteq \mathbb{C}[z_0, \ldots, z_n]$ is *homogeneous* if it can be generated by homogeneous polynomials, or equivalently, whenever $f \in I$, each homogeneous component of f is also in I.

Suppose $V \subseteq \mathbb{P}^n$ is a projective variety. Then, the set

$$\left\{ f \in \mathbb{C}[z_0, \ldots, z_n] : \forall p \in V, f(p) = 0 \right\}$$

is called the *homogeneous vanishing ideal of* V and is denoted by $\mathbb{I}(V)$.

Lemma 7.3. *$\mathbb{I}(V)$ is a homogeneous radical ideal of $\mathbb{C}[z_0, \ldots, z_n]$.*

Theorem 7.4 (Projective Nullstellensatz)**.** *There is an inclusion-reversing bijective correspondence*

$$\begin{aligned} \{\textit{projective varieties in } \mathbb{P}^n\} &\cong \{\textit{radical homogeneous ideals in } \mathbb{C}[z_0, \ldots, z_n]\} \setminus \left\{ \langle z_0, \ldots, z_n \rangle \right\} \\ V &\mapsto \mathbb{I}(V) \\ \mathbb{V}(I) &\leftarrow\!\shortmid I \end{aligned}$$

Given a projective variety $V \subseteq \mathbb{P}^n$, its *homogeneous coordinate ring* is the quotient

$$\frac{\mathbb{C}[z_0, \ldots, z_n]}{\mathbb{I}(V)}$$

Note that this is equal to the *affine* coordinate ring of the affine cone over V. Elements of this ring are also *not* functions on V, and are instead functions on the affine cone over V. This ring also depends on the embedding of V in $\mathbb{P}^n$, and not just the isomorphism class of V.

Given any subset $X \subseteq \mathbb{P}^n$, X inherits the Zariski topology as the subspace topology, where closed subsets of X are sets of the form $X \cap V$ for $V \subseteq \mathbb{P}^n$ a projective variety.

7.2 Homogenisation

Given any polynomial $f \in \mathbb{C}[x_1, \dots, x_n]$ of degree d in n variables, we can *homogenise* the polynomial into a homogeneous polynomial $f^+ \in \mathbb{C}[z_0, \dots, z_n]$ of degree d in $n+1$ variables by "padding" lower degree components with a new variable.

Decomposing f into its homogeneous components,

$$f = f_0 + f_1 + \cdots + f_{d-1} + f_d$$

the homogenisation f^+ is given by multiplying each f_i by z_0^{d-i}, and replacing each x_i with z_i:

$$f^+ = z_0^d f_0 + z_0^{d-1} f_1 + \cdots + z_0 f_{d-1} + f_d$$

Example. The polynomial $2 + 3x_1 + 4x_1^2 x_2 + 5x_2^3$ is degree 3. The homogenisation is given by

$$2z_0^3 + 3z_0^2 z_1 + 4z_1^2 z_2 + 5z_2^3$$

△

Equivalently, we can replace each x_i with $\frac{z_i}{z_0}$, then multiply through by z_0^d.

We can also *dehomogenise* a homogeneous polynomial $f \in \mathbb{C}[z_0, \dots, z_n]$ with respect to a variable z_i to obtain a polynomial in $f^\circ \in \mathbb{C}[x_1, \dots, x_n]$ by evaluating $f(z_0, \dots, z_n)$ at $(x_1, \dots, z_i = 1, \dots, x_n)$. This corresponds to restricting the homogenised polynomial to the affine chart $\mathcal{U}_i$.

The dehomogenisation of a degree d homogeneous polynomial f (with respect to z_0 for example) can be seen as two steps:

$$f(z_0, \dots, z_n) \overset{\substack{\text{divide} \\ \text{by } z_0^d}}{\mapsto} \frac{f(z_0, \dots, z_n)}{z_0^d} \overset{x_i = \frac{z_i}{z_0}}{\mapsto} \text{polynomial in } x_1, \dots, x_n$$

Thus, the following are equivalent:

- $[z_0 : \cdots : z_n] \in \mathbb{V}(f) \cap \mathcal{U}_0$;
- $(\frac{z_1}{z_0}, \dots, \frac{z_n}{z_0}) = (x_1, \dots, z_n) \in \mathbb{A}^n$;
- $f(z_0, \dots, z_n) = 0$ and $z_0 \neq 0$;
- $\frac{1}{z_0^d} f(z_0, \dots, z_n) = 0$;
- $f(1, \frac{z_1}{z_0}, \dots, \frac{z_n}{z_0}) = 0$;
- $f(1, x_1, \dots, x_n) = 0$;
- $f^\circ(x_1, \dots, x_n) = 0$.

So, there is a bijection

$$\mathbb{V}(f) \cap \mathcal{U}_0 \xrightarrow[\cong]{\psi_0} \mathbb{V}(\text{dehomogenisation of } f \text{ with respect to } z_0)$$

Lemma 7.5. *If $\{F_\alpha\}$ is a set of homogeneous polynomials in $z_0, \dots, z_n$ and f_α is the dehomogenisation of F_α with respect to z_i, then*

$$\psi_i : \mathbb{V}(\{F_i\}) \cap \mathcal{U}_i \to \mathbb{V}(\{f_i\})$$

is a bijection.

Corollary 7.5.1. *If $V \subseteq \mathbb{P}^n$ is a projective subvariety, then $V \cap \mathcal{U}_i \subseteq \mathcal{U} \cong \mathbb{A}^n$ is an affine subvariety of $\mathbb{A}^n$ under the identification $\psi_i : \mathcal{U}_i \xrightarrow{\cong} \mathbb{A}^n$.*

Lemma 7.6. *For any $f \in \mathbb{C}[x_1, \dots, x_n]$,*

$$(f^+)^\circ = f$$

where the dehomogenisation is with respect to z_0.

Corollary 7.6.1. *Given $\{f_\alpha\} \subseteq \mathbb{C}[x_1, \dots, x_n]$ and $W = \mathbb{V}(\{f_\alpha\}) \subseteq \mathbb{A}^n$, define $V = \mathbb{V}(\{f_\alpha^+\}) \subseteq \mathbb{P}^n$. Then,*

$$\psi_0(V \cap \mathcal{U}_0) = W$$

So, the affine subvarieties W of $\mathcal{U}_0 \cong \mathbb{A}^n$ are precisely the sets $V \cap \mathcal{U}_0$, where $V \subseteq \mathbb{P}^n$ is a projective subvariety. In other words, the Zariski topology on $\mathcal{U}_0$ as a subspace topology of $\mathbb{P}^n$ is the same as the Zariski topology on $\mathbb{U}_0 \cong \mathbb{A}^n$.

Note that the dehomogenisation map $\{\text{homogeneous polynomials in} z_0, \dots, z_n\} \to \mathbb{C}[x_1, \dots, x_n]$ is not injective, since

$$(z_i^k F)^\circ = F^\circ$$

where the dehomogenisation is with respect to z_i; extra factors of the new variable are discarded under dehomogenisation.

Lemma 7.7. *Let $F \in \mathbb{C}[z_0, \dots, z_n]$ be a homogeneous polynomial, and suppose $F = z_0^k G$ where $z_0 \nmid G$. Then,*

$$(F^\circ)^+ = G$$

Corollary 7.7.1. *Two homogeneous functions F_1 and F_2 have equal dehomogenisations with respect to z_0 if an only if there exists a polynomial G such that $z_0 \nmid G$ and $F_1 = z_0^k G$ and $F_2 = z_0^\ell G$ for some $k, \ell \in \mathbb{N}$.*

7.3 Projective Closures

Let V be an affine variety, with a fixed embedding $V \subseteq \mathbb{A}^n \subseteq \mathbb{P}^n$. The *projective closure* $\overline{V}$ of V is the closure of V in $\mathbb{P}^n$. The closure may be computed in either the Zariski or standard topology on $\mathbb{P}^n$; the result will be the same.

Given an affine variety $V = \mathbb{V}(F_1, \dots, F_r) \subseteq \mathbb{A}^n$, we'd might think that the projective closure $\overline{V}$ of V in $\mathbb{P}^n$ might be defined by the ideal obtained by replacing each of the polynomials F_i with its homogenisation F^+.

Example. Consider the parabola $V = \mathbb{V}(y - x^2) \subseteq \mathbb{A}^2 \subseteq \mathbb{P}^2$. The variables x and y are the affine coordinates for V in $\mathbb{A}^2$, while in $\mathbb{P}^2$, we use homogeneous coordinates x, y, and z, and identify $\mathbb{A}^2$ with the open affine chart $\mathcal{U}_z$ where z is non-zero (say, $z = 1$).

The points of the parabola in $\mathbb{P}^2$ are then the lines through the origin in $\mathbb{C}^3$ connecting to the points on the parabola in the plane $z = 1$, i.e. picture the affine cone modulo scaling. There is a line "missing" from this cone – namely the y-axis where the two branches of the parabola asymptotically converge together.

As a projective variety, the parabola is described by $yz - x^2$ in $\mathbb{P}^2$, so the projective closure of $y - x^2$ is $yz - x^2$. △

In this case, the closure is indeed given by the homogenisation. However, this does not work in general.

Example. Let $T = \mathbb{V}(y - x^2, z - xy) \subseteq \mathbb{A}^n$. We have $z - xy = z - x^3$, so the points of this variety are of the form (x, x^2, x^3), so T is the *twisted cubic*, i.e. the image of the map $\mathbb{A}^1_t \to \mathbb{A}^3_{x,y,z} : t \mapsto (t, t^2, t^3)$. What is the projective closure of T?

The homogenisations of the polynomials are given $wy - x^2$ and $wz - xy$. Let $V = \mathbb{V}(wy - x^2, wz - xy) \subseteq \mathbb{P}^3$. By construction, $V \cap \mathcal{U}_w = T$ (i.e. set $w = 1$). What about $V \setminus T$?

First,

$$V \setminus T = V \setminus (V \cap \mathcal{U}_w) = V \setminus \mathcal{U}_w = V \cap \mathbb{V}(w)$$

so, setting $w = 0$ in the defining polynomials for V, we have $wy - x^2 = -x^2 = 0$ and $wz - xy = -xy = 0$, so $x = 0$.

So, V also contains the point $[0 : y : z : 0]$ where $x = 0 = w$. However, the set $W = T \cup \{[0 : 0 : 1 : 0]\}$ is a closed set strictly contained in V, so V is not the minimal closed set containing T, i.e., $V \neq \overline{T}$. △

Theorem 7.8. *Let $V \subseteq \mathbb{A}^n \subseteq \mathbb{P}^n$ be an affine variety, and let $I = \mathbb{I}(V) \subseteq \mathbb{C}[x_1, \ldots, x_n]$ be the radical ideal of all polynomials vanishing on V. Then, the ideal $J = \langle f^+ \mid f \in I \rangle$ of $\mathbb{C}[z_0, \ldots, z_n]$ generated by the homogenisations of all the elements of I is the radical homogeneous ideal of polynomials vanishing on the projective closure $\overline{V}$ in $\mathbb{P}^n$.*

The ideal J is called the *homogenisation* of the ideal I.

The problem in the previous example is that we only homogenised the polynomials $y - x^2$ and $z - xy$. The previous theorem says that if we instead homogenise all the polynomials in the ideal closure of $y - x^2$ and $z - xy$, the generated ideal would then correspond to the projective closure.

7.4 Morphisms of Projective Varieties

Consider the map $f : \mathbb{P}^1_{[s:t]} \to \mathbb{P}^2_{[x:y:z]}$ defined by $[s : t] \mapsto [s^2 : st : t^2]$.

This map is well-defined since

$$[s : t] = [\lambda s : \lambda t] \mapsto [\lambda^2 s^2 : \lambda^2 st : \lambda^2 t^2] = [s^2 : st : t^2]$$

and since $[s : t] \in \mathbb{P}^1_{[s:t]}$, s and t cannot simultaneously vanish, so the first and last coordinate of $[s^2 : st : t^2]$ cannot simultaneously vanish, so f does not map onto the origin. More generally, any map between projective spaces is well-defined if it is given in coordinates by homogeneous polynomials of the same degree with empty common vanishing loci.

Since in the image of f, we have $x = s^2$, $y = st$, and $z = t^2$, the coordinates satisfy the relation $xz = y^2$, so the image of f lies on the curve $C = \mathbb{V}(xz - y^2)$ in $\mathbb{P}^2$. Let us examine f on affine charts.

If $s \neq 0$, then $x = s^2 \neq 0$, so $f|_{\mathcal{U}_s} \subseteq \mathcal{U}_x$. Similarly, $f|_{\mathcal{U}_t} \subseteq \mathcal{U}_z$.

On $\mathcal{U}_s$, we have $s \neq 0$, so $x = s^2 \neq 0$ and $f|_{\mathcal{U}_s} \subseteq \mathcal{U}_x$:

$$\begin{aligned} f|_{\mathcal{U}_s} : \mathcal{U}_s &\to \mathcal{U}_x \\ [s : t] = [1 : \tfrac{t}{s}] &\mapsto [1 : \tfrac{t}{s}, \tfrac{t^2}{s^2}] \end{aligned}$$

Identifying $\mathcal{U}_s$ with $\mathbb{A}^1_a$ and $\mathcal{U}_x$ with $\mathbb{A}^2_{u_1,u_2}$, this map is:

$$\begin{aligned} f|_{\mathcal{U}_s} : \mathbb{A}^1_a &\to \mathbb{A}^2_{u_1,u_2} \\ a &\mapsto (a,a^2) \end{aligned}$$

This is a morphism of affine varieties, whose image is the parabola $\mathbb{V}(u_2 - u_1^2) \cong C \cap \mathcal{U}_x$ in the plane.

Similarly, on $\mathcal{U}_t$, $f|_{\mathcal{U}_t} : \mathbb{A}^1_b \to \mathbb{A}^2_{v_1,v_2}$ is described by $b \mapsto (b^2,b)$. Again, the image is a parabola $\mathbb{V}(v_2^2 - v_1)$ in the plane.

Thus, $f : \mathbb{P}^1 \to C$ restricts localy on the coordinate charts covering $\mathbb{P}^1$ to a morphism of affine varieties. This motivates the following definition.

Let $V \subseteq \mathbb{P}^n$ and $W \subseteq \mathbb{P}^m$ be projective varieties. A map of sets $F : V \to W$ is a *morphism of projective varieties* if F is locally a polynomial map at every point of V. That is, for each $p \in V$, there exists an open neighbourhood $U \subseteq V$ of p and homogeneous polynomials $F_0, \ldots, F_m \in \mathbb{C}[z_0, \ldots, z_n]$ such that

- The F_i do not simultaneously vanish on U;
- The restriction $F|_U : U \to W$ agrees with the map $U \to \mathbb{P}^m$ defined by:

$$[z_0 : \cdots : z_n] \mapsto \big[F_0(z_0, \ldots, z_n) : F_1(z_0, \ldots, z_n) : \cdots : F_m(z_0, \ldots, z_n)\big]$$

A morphism of projective varieties is, as usual, an *isomorphism* if it has an map that is also a morphism. Two projective varieties are *isomorphic* if there exists an isomorphism between them.

This definition is compatible with the definition of morphisms for affine varieties.

Example. The simplest example of a isomorphism is given by a change of coordinates in $\mathbb{P}^n$. Let $A = (A_{ij})$ be a full-rank $(n+1)\times(n+1)$ matrix. Then, $A : \mathbb{C}^{n+1} \to \mathbb{C}^{n+1}$ is a linear automorphism, and permutes the 1-dimensional subspaces of $\mathbb{C}^{n+1}$, thus inducing an automorphism $\mathbb{P}^n \to \mathbb{P}^n$:

$$[z_0 : \ldots : z_n] \mapsto \left[\sum_j A_{0j} z_j : \cdots : \sum_j A_{nj} z_j\right]$$

△

Theorem 7.9. *Every automorphism of $\mathbb{P}^n$ arises this way, i.e. is a linear automorphism.*

Corollary 7.9.1. *If $\lambda \in \mathbb{C}$ is a non-zero scalar, then A and λA induce the same automorphism of $\mathbb{P}^n$.*

Two projective varieties $V, W \subseteq \mathbb{P}^n$ are *projectively equivalent* if there exists an automorphism of $\mathbb{P}^n$ that restricts to an isomorphism $V \to W$.

Theorem 7.10. *If V and W are projectively equivalent, then their homogeneous coordinate rings are isomorphic.*

Theorem 7.11. *If $F, G \in \mathbb{C}[x,y,z]$ are irreducible homogeneous polynomials of degree 2, then $\mathbb{V}(F) \subseteq \mathbb{P}^2$ and $\mathbb{V}(G) \subseteq \mathbb{P}^2$ are projectively equivalent.*

Let $A = (A_{ij})$ be an $(m+1)\times(n+1)$ matrix with trivial nullspace (so $m \geq n$), representing an injective linear map $\mathbb{C}^{n+1} \to \mathbb{C}^{m+1}$. This induces a morphism $\mathbb{P}^n \to \mathbb{P}^m$ that is linear in homogeneous coordinates.

This map is an embedding, i.e., is an isomorphism on to a closed subvariety of $\mathbb{P}^m$, and the image is a linear subvariety, i.e. is cut out by homogeneous polynomials of degree 1.

Theorem 7.12. *If $n > m$, then there do not exist any non-constant morphisms $\mathbb{P}^n \to \mathbb{P}^m$. If $n \leq m$, then there exist non-linear morphisms $\mathbb{P}^n \to \mathbb{P}^m$.*

8 Quasiprojective Varieties

A *locally closed* subset of a topological space X is the intersection of an open and closed subset, or equivalently, a closed subset of an open subset.

A *quasiprojective variety* is a locally closed subset of $\mathbb{P}^n$. A quasiprojective variety inherits the Zariski topology from $\mathbb{P}^n$.

Example. The following are quasiprojective:

- $\mathbb{P}^n = \mathbb{P}^n \cap \mathbb{P}^n$;
- $\mathbb{A}^n = \mathcal{U}_0 \cap \mathbb{P}^n$;
- Any projective variety $W \subseteq \mathbb{P}^n$, $W = \mathbb{P}^n \cap W$;
- Any closed affine variety $V \subseteq \mathbb{A}^n$, since any affine variety can be viewed as an open subset of its affine cone, $V = \mathcal{U}_0 \cap \overline{V}$.

- Any open set $X \subseteq \mathbb{P}^n$, since $X = X \cap \mathbb{P}^n$;
- Any open set $Y \subseteq \mathbb{A}^n$, since $Y \subseteq \mathbb{P}^n$ is also open.
- $U = \mathbb{A}^1_t \setminus \{0\}$ is a quasiprojective variety. To see this, embed X into $\mathcal{U}_s \subseteq \mathbb{P}^1_{[t:s]}$, i.e. along the line $s = 1$ via $t \mapsto [t : 1]$. Because U is missing the origin, we remove $[0 : 1]$ from $\mathbb{P}^1$, and also, the point at infinity $[1 : 0]$ has no preimage in $\mathbb{A}^1$, so it too is removed. So $U = \mathbb{P}^1_{[t:s]} \setminus \big\{[0 : 1],[1 : 0]\big\}$ is open in $\mathbb{P}^1$, so $U = U \cup \mathbb{P}^1$ is locally closed.

△

The definition of a morphism of quasiprojective varieties is the same as for projective varieties:

Let $X \subseteq \mathbb{P}^n$ and $Y \subseteq \mathbb{P}^m$ be quasiprojective varieties. A map of sets $F : X \to Y$ is a *morphism of quasiprojective varieties* if F is locally a polynomial map at every point of V. That is, for each $p \in V$, there exists an open neighbourhood $U \subseteq X$ of p and homogeneous polynomials $F_0, \ldots ,F_m \in \mathbb{C}[z_0, \ldots ,z_n]$ such that

- The F_i do not simultaneously vanish on U;
- The restriction $F|_U : U \to W$ agrees with the map $U \to \mathbb{P}^m$ defined by:

$$[z_0 : \cdots : z_n] \mapsto \big[F_0(z_0, \ldots ,z_n) : F_1(z_0, \ldots ,z_n) : \cdots : F_m(z_0, \ldots ,z_n)\big]$$

Example. Let $X = \mathbb{A}^1_t \setminus \{0\}$ and $Y = \mathbb{V}(xy - 1) \subseteq \mathbb{A}^2_{x,y} \cong \mathbb{V}(xy - z^2) \cap \mathcal{U}_z \subseteq \mathbb{P}^2$. Both X and Y are quasiprojective varieties, and we have a well-defined map

$$\begin{aligned} F : X &\to Y \\ t &\mapsto (t,\tfrac{1}{t}) \end{aligned}$$

We claim that this is a morphism of quasiprojective varieties. To see this, we embed X into $\mathbb{P}^1$ via $t \mapsto [t : 1]$ (as before), and Y into $\mathbb{P}^2$ via $(x,y) \mapsto [x : y : 1]$ (since z does not vanish in $\mathbb{V}(xy - z^2) \cap \mathcal{U}_z$). Then, F agrees everywhere on U with the morphism

$$\begin{aligned} \tilde{F} : \mathbb{P}^1 &\to \mathbb{P}^2 \\ [t : s] &\mapsto [t^2 : s^2 : st] \end{aligned}$$

On $U = \mathbb{P}^1_{[t:s]} \setminus \big\{[0 : 1],[1 : 0]\big\}$, neither t nor s vanish, so setting $t = a/b$, we see

$$U \ni t = [t : 1] \overset{\tilde{F}}{\mapsto} [t^2 : 1 : t] = [t : \tfrac{1}{t} : 1] = (t,\tfrac{1}{t}) \in Y$$

which agrees with F. △

Every morphism of projective varieties is a morphism of quasiprojective varieties, since the definition is effectively identical, but also every morphism of affine varieties is a morphism of quasiprojective varieties.

Having defined quasiprojective varieties, we now redefine the concept of an affine variety.

A quasiprojective variety is *affine* if it is isomorphic to a closed subset of affine space, i.e. to a subvariety of $\mathbb{A}^n$.

Example. The open set $X = \mathbb{A}^1 \setminus \{0\} \subseteq \mathbb{A}^1_t$ is an affine variety, because it is isomorphic as a quasiprojective variety to $Y = \mathbb{V}(xy - 1) \subseteq \mathbb{A}^2_{x,y}$: the projection map $G : X \to Y : (x,y) \mapsto x$ is a morphism of quasiprojective varieties and is the inverse of the map $F : U \to V$ defined above. △

The *coordinate ring* $\mathbb{C}[X]$ of an affine quasiprojective variety is the $\mathbb{C}$-algebra $\mathbb{C}[V]$, where $V \subseteq \mathbb{A}^n$ is closed (i.e. is a affine subvariety) and $X \cong V$ as quasiprojective varieties. That is, if $F : X \to V$ is an isomorphism, then the coordinate ring $\mathbb{C}[X]$ is the ring of functions $W \to \mathbb{C}$ that are pullbacks of functions in $\mathbb{C}[V]$.

Example. $\mathbb{C}\big[\mathbb{A}^1 \setminus \{0\}\big] = \frac{\mathbb{C}[x,y]}{\langle xy-1\rangle} \cong \mathbb{C}[t,t^{-1}]$. △

Similarly, a quasiprojective variety is *projective* if it is isomorphic to a closed subset of projective space, i.e. to a subvariety of $\mathbb{P}^n$. Unlike the case for affine varieties, this redefinition does not enlarge the class of projective varieties.

Theorem 8.1. *If X is both affine and projective, then X is isomorphic to a finite set of points.*

Theorem 8.2. *If $X \subseteq \mathbb{P}^n$ is a quasiprojective variety and there exists a closed set $Y \subseteq \mathbb{P}^m$ with $X \cong Y$ as quasiprojective varieties, then X is closed in $\mathbb{P}^n$.*

8.1 Quasiprojective Varieties are Locally Affine

The Zariski topology for any quasiprojective varieties has a basis of open affine sets.This allows us to think of every quasiprojective variety as "locally affine", in the same way that every manifold is "locally Euclidean". That is, each point in a quasiprojective variety has an open neighbourhood that is an affine subvariety.

First, observe that the complement of any hypersurface in an affine variety is again an affine variety. Specifically, if V is a Zariski-closed subset of $\mathbb{A}^n$ and $f \in \mathbb{C}[V]$, then the open set $U = V \setminus \mathbb{V}(f)$ is an affine variety (though not usually a closed set/affine subvariety of V). To see this, consider the map

$$\begin{aligned} F : U &\to \mathbb{A}^{n+1}_{x_1,\ldots,x_n,z} \\ (x_1,\ldots,x_n) &\mapsto \left(x_1,\ldots,x_n,\frac{1}{f(x_1,\ldots,x_n)}\right) \end{aligned}$$

Since f does not vanish on U by definition, this map is well-defined. Moreover, if $x_1,\ldots,x_n,z$ denote the coordinates for $\mathbb{A}^{n+1}$, the original defining polynomials $F_1(x_1,\ldots,x_n),\ldots,F_r(x_1,\ldots,x_n)$ for V in $\mathbb{A}^n$ all vanish at the image points of F, as does the polynomial $zf(x_1,\ldots,x_n) - 1$. So, the image of F is contained in the Zariski-closed subset of $\mathbb{A}^{n+1}$ defined by $W = \mathbb{V}(F_1,\ldots,F_r,zf-1)$, and the map $U \to \mathbb{V}(F_1,\ldots,F_r,zf-1) \subseteq \mathbb{A}^{n+1}$ is an isomorphism of quasiprojective varieties. So, $V \setminus \mathbb{V}(g) \cong W$ is an affine quasiprojective variety.

Lemma 8.3. *The open sets of the form*

$$V \setminus \mathbb{V}(g)$$

where $g \in \mathbb{C}[V]$ is non-zero and non-unit form a basis for the Zariski topology on V.

These sets are called *basic affine open sets* .

Theorem 8.4. *There is a basis for the Zariski topology on every quasiprojective variety $V \subseteq \mathbb{P}^n$ consisting of basic affine open sets.*

Corollary 8.4.1. *Quasiprojective varieties are also locally affine.*

8.2 Regular Functions

Regulan functions are the generalisation of polynomial functions on affine varieties to the case of quasiprojective varieties.

While manifolds locally look like Euclidean space $\mathbb{R}^n$, quasiprojective varieties locally look like affine varieties. The existence of a basis of basic affine open sets means that we can view every variety as a union of affine varieties, and so we can define a regular function locally as a function that restricts on each affine patch to a polynomial function.

Let V be a Zariski-closed subset of $\mathbb{A}^n$. Given $f,g \in \mathbb{C}[V]$, the rational expression $\frac{f}{g}$ is locally well-defined on $V \setminus \mathbb{V}(g)$.

Theorem 8.5. *If $W \cong V \setminus \mathbb{V}(g)$, then*

$$\mathbb{C}[W] \cong \mathbb{C}[V][\tfrac{1}{g}] \cong \frac{\mathbb{C}[V][z]}{\langle zg - 1 \rangle}$$

On the chart $V \setminus \mathbb{V}(g)$, the function $\frac{1}{g}$ is identified with the polynomial z on $\mathbb{A}^{n+1}$, and the function $\frac{f}{g}$ is identified with the polynomial zf on $\mathbb{A}^{n+1}$. We now extend this definition to affine varieties that are not necessarily closed in $\mathbb{A}^{n+1}$.

Let U be any open subset of a Zariski-closed subset V of affine space. A function $F :\to \mathbb{C}$ is *regular at* $p \in U$ if there exist $f,g \in \mathbb{C}[V]$ such that

- $g(p) \neq 0$;
- there exists an open neighbourhood $W \subseteq U$ of p such that g is non-zero on W, and $F|_W = \frac{f}{g}$.

F is *regular on* U if it is regular at every point $p \in U$.

Example. The slope function

$$\begin{aligned} f : U = \mathbb{A}^2 \setminus \mathbb{V}(x) &\to \mathbb{C} \\ (x,y) &\mapsto \frac{y}{x} \end{aligned}$$

is regular on U. △

We define $\mathcal{O}_V(U)$ to be the set of regular functions $U \to \mathbb{C}$:

$$\mathcal{O}_V(U) = \{F : U \to \mathbb{C} : F \text{ regular}\} \subseteq \mathbb{C}[V]$$

This set is a $\mathbb{C}$-algebra under pointwise addition, multiplication, and scaling of functions.

Example. Let $X = \mathbb{A}^2$ and $Y = \mathbb{A}^2 \setminus \mathbb{V}(x)$. Then, $\frac{1}{x}, \frac{y}{x} \in \mathcal{O}_V(U) = \mathbb{C}[x,y,\frac{1}{x}]$. △

Note that the restriction $\mathbb{C}[V] \to \mathcal{O}_V(U)$ is injective if U is dense in V; in particular, if V is irreducible and U is non-empty.

Theorem 8.6. *The inclusion $\mathbb{C}[V] \hookrightarrow \mathcal{O}_V(V)$ is surjective and is hence an isomorphism.*

This is non-obvious, saying that every locally rational function is in fact globally polynomial.

Theorem 8.7. *Let $W = V \setminus \mathbb{V}(h)$ be a basic affine open set, and let $U \subseteq W$ be open, not necessarily affine. Then,*

$$\mathcal{O}_V(U) = \mathcal{O}_W(U)$$

We now generalise $\mathcal{O}_V(U)$ from affine V to quasiprojective V. Given a quasiprojective variety V and an open subset $U \subseteq V$,

$$\mathcal{O}_V(U) = \left\{F : U \to \mathbb{C} : \forall p \in U, \text{open neighbourhood } W \subseteq U \text{ of } p \text{ such that } F|_W \in \mathbb{C}[W]\right\}$$

Again, this is naturally a $\mathbb{C}$-algebra.

The definition of a morphism of quasiprojective varieties can also be rephrased locally using regular functions.

Let $X \subseteq \mathbb{P}^n$ and $Y \subseteq \mathbb{P}^m$ be quasiprojective varieties. A map of sets $F : X \to Y$ is a morphism of quasiprojective varieties if for each $p \in V$, there exist open affine neighbourhoods U of p and V of $f(p)$ such that $f(U) \subseteq V$ and $f|_U$ agrees with a map of affine varieties. That is, $f|_U$ is given by a set of regular functions in the coordinates of U.

9 The Veronese Embedding

The Veronese embedding is an important example of a morphism of quasiprojective varieties. The Veronese embedding embeds $\mathbb{P}^n$ as a subvariety of a higher dimensional projective space in a non-trivial way.

Consider the set of all homogeneous polynomials of fixed degree d in the polynomial ring $\mathbb{C}[x_0, \ldots, x_n]$. This is a finite-dimensional $\mathbb{C}$-vector space, with standard basis given by the monomials of the form

$$\prod_{i=1}^{d} x_i^{d_i}$$

where $\sum_{i=1}^{d} d_i = d$. We define the set of exponent vectors

$$M_{d,n} \coloneqq \left\{ I = (d_0, \ldots, d_n) \in \mathbb{N}^{n+1} : \sum_{i=1}^{n} d_i = d \right\} \cong \{\text{degree } d \text{ monomials in } x_0, \ldots, x_n\}$$

with the obvious bijection sending each vector $I = (d_0, \ldots, d_n)$ to the monomial $x_0^{d_0} \cdots x_n^{d_n}$. We abbreviate thes monomial to x^I.

Example. If $n = 2$ and $d = 6$, then the vector $I = (0,2,4)$ corresponds to the monomial $x_0^0 x_1^2 x_3^4$. △

These vectors (and hence monomials, by transport along the bijection) are naturally ordered under the lexicographic ordering, where $I = (i_0, \ldots, i_n)$ precedes $J = (j_0, \ldots, j_n)$ if there exists $k \in \mathbb{N}$ such that $i_k < j_k$, and $i_\ell = j_\ell$ for all $\ell < k$. That is, the first disagreement between I and J in the kth

Example. $x^I \coloneqq x_0^2 x_1 x_2^3 x_4$ precedes $x^J \coloneqq x_0^2 x_1 x_2^2, x_3^2$ since

$$I = (2,1,3,0,1) > (2,1,2,2,0) = J$$

△

Theorem 9.1. *There are $\binom{n+d}{d}$-many degree d monomials in $n+1$ variables $x_0, \ldots, x_n$.*

Proof. Stars and bars. Every monomial can be represented by a string of x_1-many stars, a separating bar, x_2-many stars, etc. of d stars and n separating bars, and there are $\binom{n+d}{d}$ ways to place the d stars amongst the $d+n$ total spaces. ■

The dth Veronese embedding of $\mathbb{P}^n$ is the map $\nu_{d,n}$ defined by the tuple of all monomials of degree d:

$$\begin{aligned} \nu_{d,n} : \mathbb{P}^n &\to \mathbb{P}^{|M_{d,n}|-1} = \mathbb{P}^{\binom{n+d}{d}-1} \\ [x_0 : \cdots : x_n] &\mapsto [\underbrace{x_0^d : x_0^{d-1} x_1 : \cdots : x_n^d}_{\text{all monomials of degree } d}] \end{aligned}$$

This is well-defined, since the polynomials all have the same degree, and cannot simultaneously vanish for any $[x_0 : \cdots : x_n] \in \mathbb{P}^n$, since if $x_i \neq 0$, then $x_i^d \neq 0$.

Example. The 2nd Veronesi embedding in dimension 1 is given by

$$\begin{aligned} \nu_{2,1} : \mathbb{P}^1_{[s:t]} &\to \mathbb{P}^2_{[x:y:z]} \\ [s : t] &\mapsto [s^2 : st : t^2] \end{aligned}$$

and is an isomorphism on to its image $\mathbb{V}(xz - y^2)$.

The 3rd Veronesi embedding in dimension 1 is given by

$$\nu_{3,1} : \mathbb{P}^1_{[s:t]} \to \mathbb{P}^2_{[x:y:z:w]}$$

$$[s:t] \mapsto [s^3 : s^2t : st^2 : t^3]$$

$\triangle$

In general, we may index the coordinates of $\mathbb{P}^{|M_{d,n}|-1}$ by $I \in M_{d,n}$.

Example. The 2nd Veronesi embedding in dimension 2 is given by

$$\begin{aligned} \nu_{2,2} : \mathbb{P}^2_{[x_0:x_1:x_2]} &\to \mathbb{P}^5_{[z_{(2,0,0)}:\cdots:z_{(0,0,2)}]} \\ [x_0 : x_1 : x_2] &\mapsto [\underbrace{x_0^2}_{z_{(2,0,0)}} : \underbrace{x_0x_1}_{z_{(1,1,0)}} : \underbrace{x_0x_2}_{z_{(1,0,1)}} : \underbrace{x_1^2}_{z_{(0,2,0)}} : \underbrace{x_1x_2}_{z_{(0,1,1)}} : \underbrace{x_2^2}_{z_{(0,0,2)}}] \end{aligned}$$

$\triangle$

Theorem 9.2. *For all d,n, the Veronesi embedding $\nu_{d,n}$ is an isomorphism from $\mathbb{P}^n$ onto a closed subvariety of $\mathbb{P}^{|M_{d,n}|-1}$.*

Proof. We describe the inverse map.

Let $W \subseteq \mathbb{P}^{\binom{n+d}{d}-1}$ be the image of $\nu_{d,n}$. At each point of W, at least one of the coordinates indexed by the single-variable monomials x_i^d must be non-zero. So,

$$\nu_{d,n}(\mathcal{U}_i) \subseteq \mathcal{U}_{(0,\dots,\underbrace{1}_{i\text{th position}},\dots,0)} \subseteq \mathbb{P}^{\binom{n+d}{d}-1}$$

for each i, where $\mathcal{U}_i$ is the subset of $\mathbb{P}^n$ where x_i is non-zero.

Also, for each i,

$$[x_0x_i^{d-1} : x_1x_i^{d-1} : \cdots : x_i^d : \cdots : x_i^{d-1}x_{n-1} : x_i^{d-1}x_n] = [x_0 : \cdots : x_n]$$

so we can define a inverse on each affine chart by:

$$\begin{aligned} \mathcal{U}_{(0,\dots,\underbrace{1}_{i\text{th position}},\dots,0)} &\to \mathbb{P}^n \\ [z_{(d,0,\dots,0)} : \cdots : z_{(0,\dots,0,d)}] &\mapsto \end{aligned}$$

$\blacksquare$